Revisiting the generalized Łoś-Tarski theorem

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Introduction

- The Łoś-Tarski theorem is a result from classical model theory characterizing FO definable extension-closed properties in terms of existential sentences.
- Historically significant:
  - constituted the earliest applications of Compactness theorem
  - set the trend for a host of preservation theorems, for not only FO but even its extensions
- The theorem fails in the finite: there is an ext.-closed FO sentence that is not equivalent to any existential sentence over all finite structures.
- This inspired investigating algorithmically important classes of structures to "recover" the theorem: classes of bounded degree, bounded tree-width, and bounded clique-width.

The Generalized Łoś-Tarski theorem

- Adsul, Chakraborty and S. proved a generalized Łoś-Tarski theorem (GLT(k)) by introducing and characterizing a parameterized generalization of ext.-closure.
- This property called k-ext.-closure semantically characterizes Π_2 sentences whose leading block contains k quantifiers.
- GLT(k) provides finer characterizations of the Π_2 class than those in the literature, further via a combinatorial notion.
- Fails over all finite structures: the semantic property evades capture by its corresponding syntactic class for each k.
- Has been recovered over various classes of posets (words, trees, nested words) and subclasses of bounded clique-width graphs (tree-depth/shrub-depth, m-partite cographs).

Main results of the paper

GLT(k) over arbitrary structures

A new and simpler proof of GLT(k) that avoids using λ-saturated structures by constructing just the "required saturation".

Łoś-Tarski theorem over finite structures

For each k, an FO sentence that is ext.-closed over all finite structures but that is not equivalent over this class to any Π_2 sentence containing k universal quantifiers.

Some notation for the talk

- $\exists^* := \exists x_1 \ldots \exists x_n \alpha(x_1, \ldots, x_n)$ for some n; α quantifier-free.
- $\forall^k \exists^* := \forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_n \alpha(x_1, \ldots, x_k, y_1, \ldots, y_n)$
- $\Pi_2 := \bigcup_{k \geq 0} \forall^k \exists^*$
- $\mathcal{A}_1 \subseteq \mathcal{A}_2 := \mathcal{A}_1$ is a substructure of $\mathcal{A}_2$.
- $\mathcal{A}_1 \equiv \mathcal{A}_2 := \mathcal{A}_1$ and $\mathcal{A}_2$ agree on all FO sentences
- $\mathcal{A}_1 \preceq \mathcal{A}_2 := \mathcal{A}_2$ is an elementary extension of $\mathcal{A}_1$.

The generalized Łoś-Tarski theorem: GLT(k)

Classical preservation properties

Definition

A sentence φ is said to be extension closed if

$$(\mathcal{A} \models \varphi \ \wedge \ \mathcal{A} \subseteq \mathcal{B}) \rightarrow \mathcal{B} \models \varphi.$$

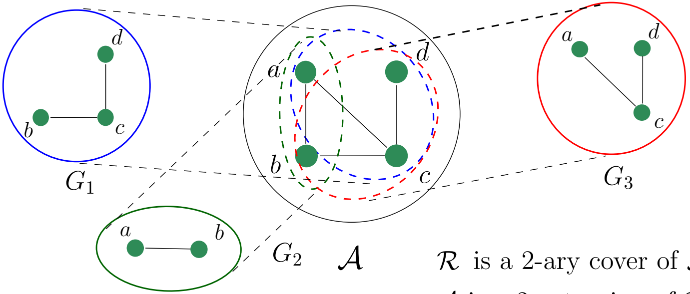
- E.g.: $\varphi :=$ "There is a triangle in the graph"'.
- Every $\exists^*$ sentence is ext.-closed.

Theorem (Łoś-Tarski, 1954-55)

Over arbitrary (finite + infinite) structures, a sentence is ext.-closed iff it is equivalent to an $\exists^*$ sentence.

Definition

For $k \in \mathbb{N}$ and a structure $\mathcal{A}$, a set $\mathcal{R}$ of substructures of $\mathcal{A}$ is called a k-ary cover of $\mathcal{A}$ if every set C of $\leq k$ elements of $\mathcal{A}$ is contained in some structure of $\mathcal{R}$. We call $\mathcal{A}$ a k-extension of $\mathcal{R}$.



$\mathcal{R}$ is a 2-ary cover of $\mathcal{A}$

$\mathcal{A}$ is a 2-extension of $\mathcal{R}$

From ext.-closure to k-ext.-closure

Definition

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Definition

A sentence φ is said to be k-extension closed if for each collection $\mathcal{R}$ of models of φ, if $\mathcal{A}$ is a k-extension of $\mathcal{R}$, then $\mathcal{A} \models \varphi$.

- Every $\forall^k \exists^*$ sentence is k-ext.-closed.

The generalized Łoś-Tarski theorem: GLT(k)

Theorem (Adsul-Chakraborty-S., 2016)

Over arbitrary (finite + infinite) structures, a sentence is k-ext. closed iff it is equivalent to a $\forall^k \exists^*$ sentence.

Proving GLT(k)

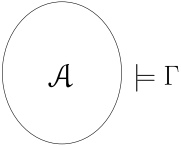
Proof outline for GLT(k)

- Given φ that is k-ext. closed, let

$$\Gamma = \{\psi \mid \psi \in \forall^k \exists^* \text{ and } \varphi \rightarrow \psi\}$$

- Then $\varphi \rightarrow \Gamma$.
- Show that $\Gamma \rightarrow \varphi$. Which implies $\varphi \leftrightarrow \Gamma$.
- By Compactness theorem, there is a finite subset Γ' of Γ such that $\varphi \leftrightarrow \Gamma'$.
- A finite conjunction of $\forall^k \exists^*$ sentences is equivalent to a $\forall^k \exists^*$ sentence. $\square$

Proof outline for GLT(k): showing $\Gamma \rightarrow \varphi$



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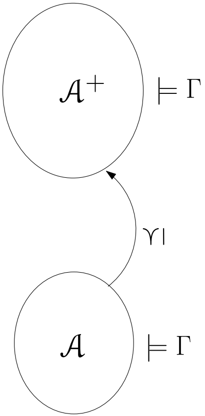

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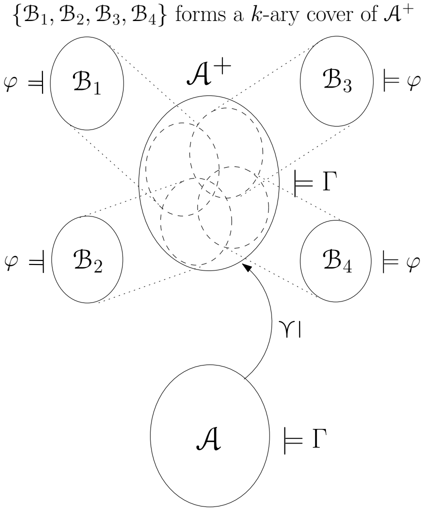

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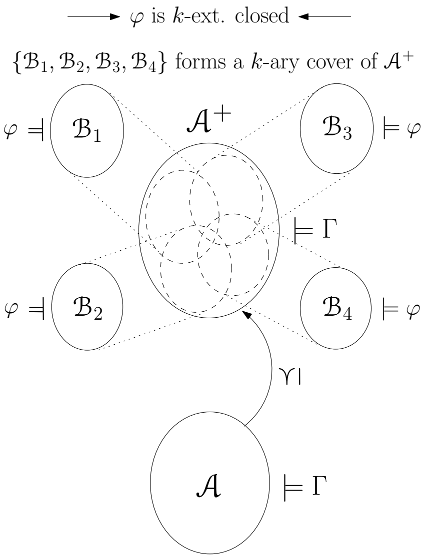
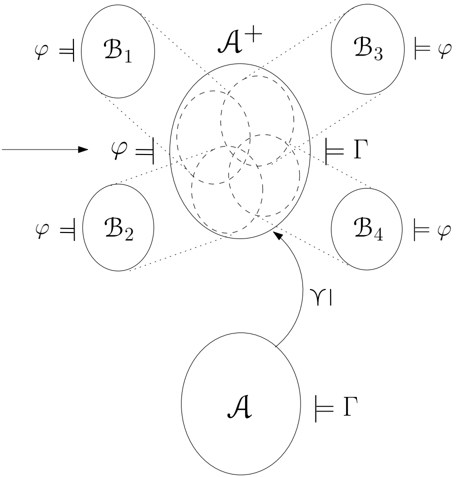


Proof outline for GLT(k): showing $\Gamma \rightarrow \varphi$

φ is k-ext. closed

$\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$ forms a k-ary cover of $\mathcal{A}^+$



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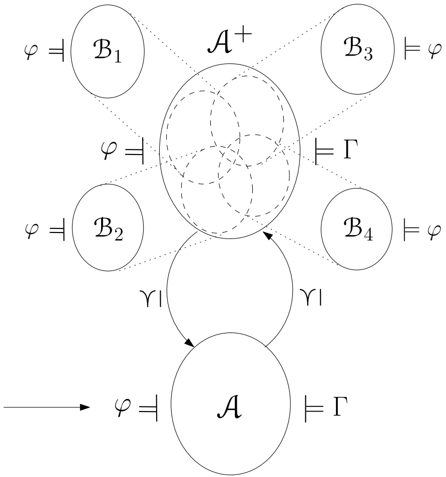
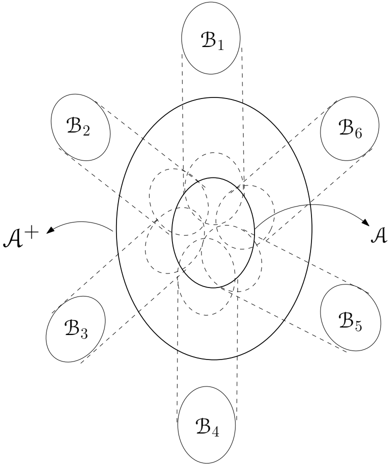


So how does one construct $\mathcal{A}^+$?

Generalized k-ary covers

- Instead of requiring "self-contained" k-ary covers, we allow for k-ary covers of a structure in an extension of it.
- In the figure alongside, each $\mathcal{B}_i \subseteq \mathcal{A}^+$ and every k-tuple of $\mathcal{A}$ is contained in some $\mathcal{B}_i$.

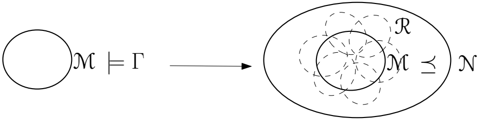


$\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6\}$ forms a k-ary cover of $\mathcal{A}$ in $\mathcal{A}^+$

Constructing $\mathcal{A}^+$ using generalized k-ary covers

Lemma (*)

For any structure $\mathcal{M}$ that models Γ, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and a k-ary cover $\mathcal{R}$ of $\mathcal{M}$ in $\mathcal{N}$, consisting of models of φ.



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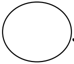

$\mathcal{A}_0 = \mathcal{A} \models \Gamma$

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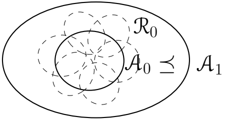

$\mathcal{R}_0$ is a k-ary cover of $\mathcal{A}_0$ in $\mathcal{A}_1$ consisting of models of φ

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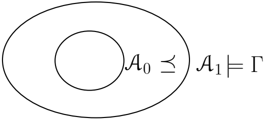

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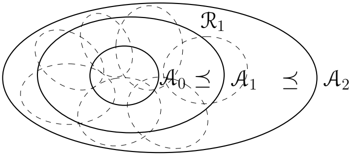

$\mathcal{R}_1$ is a k-ary cover of $\mathcal{A}_1$ in $\mathcal{A}_2$ consisting of models of φ

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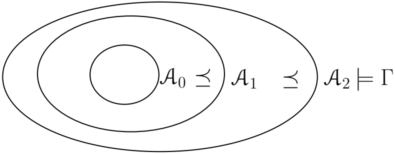

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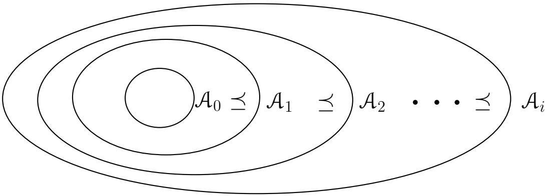

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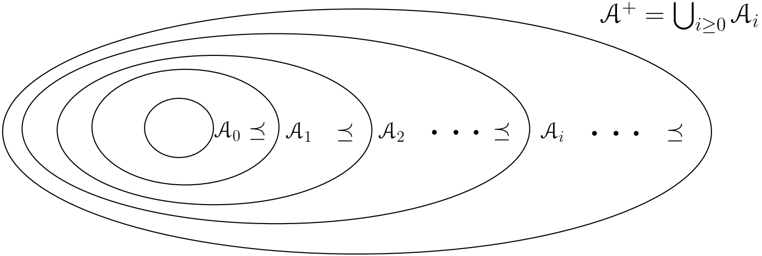

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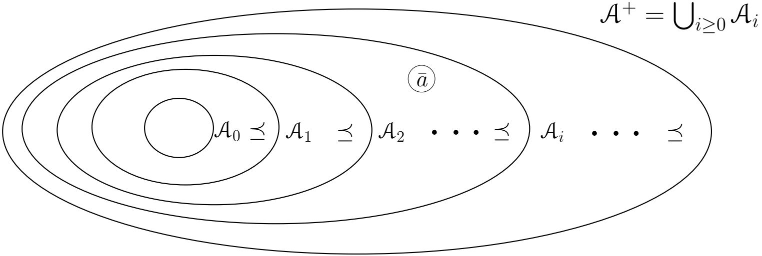

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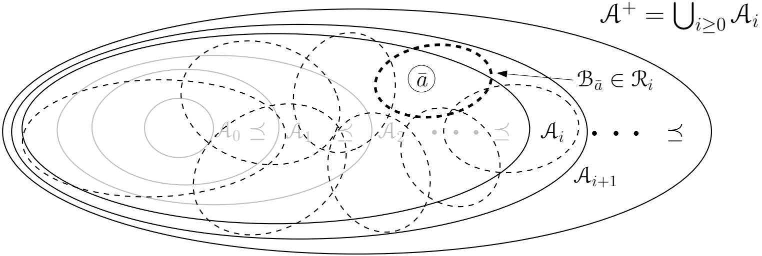

$\mathcal{R}_i$ is a k-ary cover of $\mathcal{A}_i$ in $\mathcal{A}_{i+1}$ consisting of models of φ.

Constructing $\mathcal{A}^+$ using generalized k-ary covers

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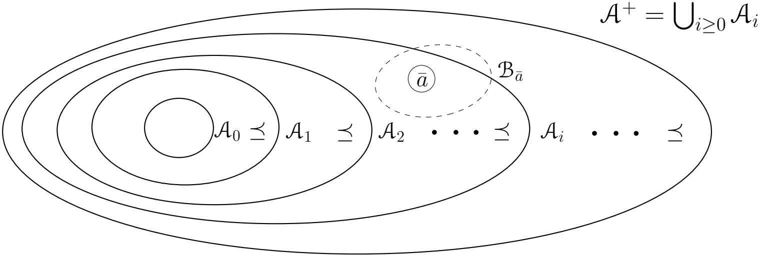


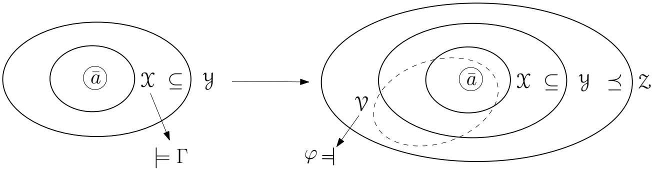
Then $\{\mathcal{B}_{\bar{a}} \mid \bar{a} \text{ is a } k \text{ tuple from } \mathcal{A}^+\}$ is a desired k-ary cover of $\mathcal{A}^+$ (in $\mathcal{A}^+$). □

Proof Sketch for Lemma (*)

Proved using transfinite induction on the k-tuples of $\mathcal{M}$, and the following simple consequence of the Compactness theorem.

Lemma (**)

Given structures $\mathcal{X}, \mathcal{Y}$ such that $\mathcal{X} \models \Gamma$ and $\mathcal{X} \subseteq \mathcal{Y}$, and a k-tuple $\bar{a} \in \mathcal{X}$, there exists $\mathcal{Z} \succeq \mathcal{Y}$ and $\mathcal{V} \subseteq \mathcal{Z}$ such that $\bar{a} \in \mathcal{V}$ and $\mathcal{V} \models \varphi$.



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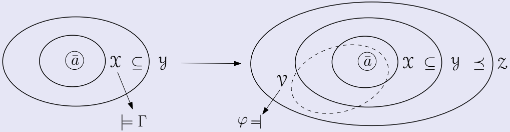

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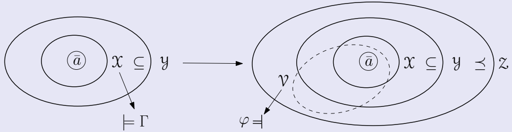

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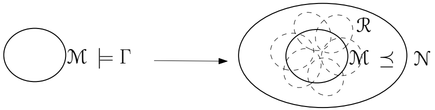

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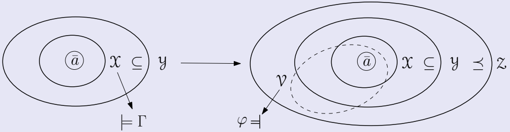

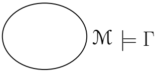

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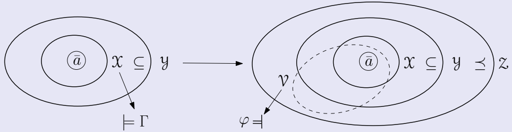

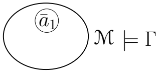

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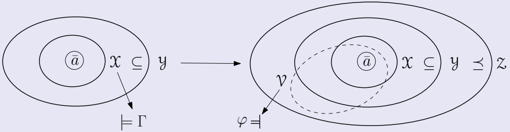

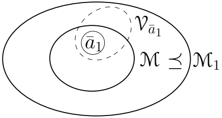

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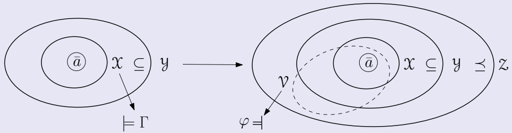

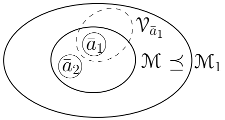

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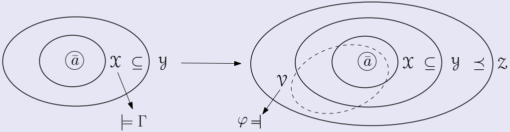

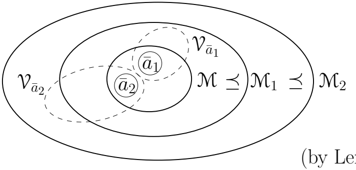

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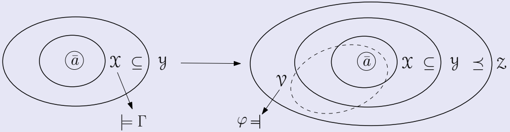

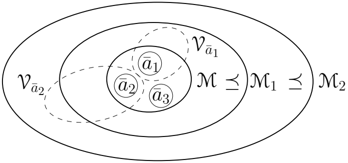

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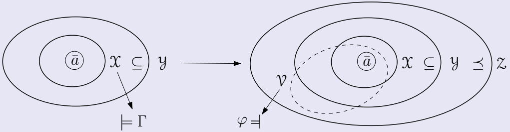

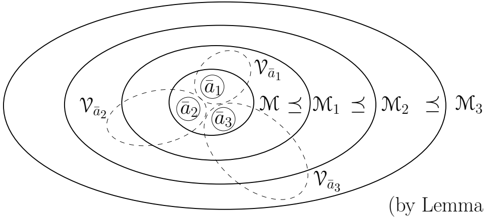

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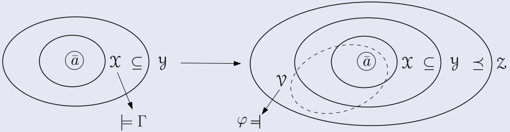

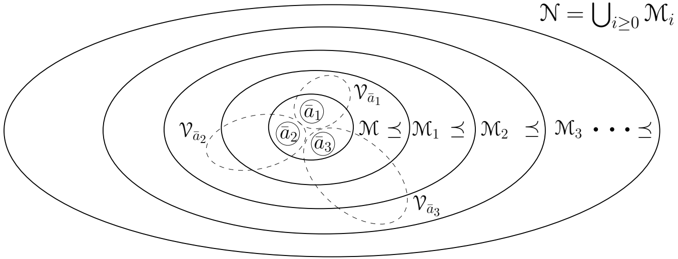

$\{\mathcal{V}_{\bar{a}} \mid \bar{a}$ is a k tuple from $\mathcal{M}\}$ is a k-ary cover of $\mathcal{M}$ in $\mathcal{N}$. □

A relook at the Łoś-Tarski theorem in the finite

Stronger failure of the Łoś-Tarski theorem in the finite

Theorem

There is a vocabulary τ such that for each $k \geq 0$, there is an FO(τ) sentence φ_k that is ext.-closed over the class $\mathcal{S}$ of all finite τ-structures, but is not equivalent over $\mathcal{S}$, to any $\forall^k \exists^*$ sentence.

Corollary

There is an FO(τ) sentence such that is k-ext.-closed over $\mathcal{S}$ but that is not equivalent over $\mathcal{S}$ to any $\forall^k \exists^*$ sentence.

Proof approach

We employ a variant of the classical Ehrenfeucht-Fräissé game as defined below for parameters k, n:

- We are given classes $\mathcal{X}$ and $\mathcal{Y}$ of structures, a single structure $\mathcal{A} \in \mathcal{X}$ and parameters k, n.

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- Round 1: The Spoiler picks a k-tuple $\bar{a}_0$ from $\mathcal{A}$. In response, the Duplicator first chooses a structure $\mathcal{B} \in \mathcal{Y}$, and then picks up a k-tuple $\bar{b}_0$ from $\mathcal{B}$.

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- Round 2: The Spoiler picks up an n-tuple $\bar{b}_1$ from $\mathcal{B}$ to which the Duplicator responds by picking an n-tuple $\bar{a}_1$ from $\mathcal{A}$.

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- Round 1: The Spoiler picks a k-tuple $\bar{a}_0$ from $\mathcal{A}$. In response, the Duplicator first chooses a structure $\mathcal{B} \in \mathcal{Y}$, and then picks up a k-tuple $\bar{b}_0$ from $\mathcal{B}$.
- Round 2: The Spoiler picks up an n-tuple $\bar{b}_1$ from $\mathcal{B}$ to which the Duplicator responds by picking an n-tuple $\bar{a}_1$ from $\mathcal{A}$.
- The Duplicator wins the above play of the game if $(\bar{a}_0, \bar{a}_1) \mapsto (\bar{b}_0, \bar{b}_1)$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Proof approach

We employ a variant of the classical Ehrenfeucht-Fräïssé game as defined below for parameters k, n:

- We are given classes $\mathcal{X}$ and $\mathcal{Y}$ of structures, a single structure $\mathcal{A} \in \mathcal{X}$ and parameters k, n.
- Round 1: The Spoiler picks a k-tuple $\bar{a}_0$ from $\mathcal{A}$. In response, the Duplicator first chooses a structure $\mathcal{B} \in \mathcal{Y}$, and then picks up a k-tuple $\bar{b}_0$ from $\mathcal{B}$.
- Round 2: The Spoiler picks up an n-tuple $\bar{b}_1$ from $\mathcal{B}$ to which the Duplicator responds by picking an n-tuple $\bar{a}_1$ from $\mathcal{A}$.
- The Duplicator wins the above play of the game if $(\bar{a}_0, \bar{a}_1) \mapsto (\bar{b}_0, \bar{b}_1)$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.
- The Duplicator has a winning strategy in the above game if she wins in every play of the game.

Proof approach (Contd.)

- Let $\exists^k\forall^n := \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_n \alpha(x_1, \ldots, x_k, y_1, \ldots, y_n)$.
- Let $\mathcal{X}, \mathcal{Y}, k$ and n be as before, and let $\mathcal{D}(\mathcal{A})$ denote the proposition that the Duplicator has a winning strategy in the described EF game started on $\mathcal{A} \in \mathcal{X}$.

Lemma

If $\mathcal{D}(\mathcal{A})$ holds, then for every $\exists^k\forall^n$ sentence ψ that is true on $\mathcal{A}$, there exists a structure $\mathcal{B} \in \mathcal{Y}$ such that ψ is true on $\mathcal{B}$ as well.

Corollary

Let φ be a given sentence, and $\xi := \neg\varphi$. Let $\mathcal{X}$ and $\mathcal{Y}$ be resp. the models and non-models of ξ. Suppose that $\mathcal{D}(\mathcal{A})$ is true for every $\mathcal{A} \in \mathcal{X}$. Then ξ cannot be equivalent to any $\exists^k\forall^n$ sentence, and hence φ cannot be equivalent to any $\forall^k\exists^n$ sentence.

Future work

An open question

Problem

Is the Łoś-Tarski theorem true over finite undirected graphs possibly with colors? The same question for directed graphs.

Dhanyavād!