Multiverse conceptions reconsidered

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Structure of the talk

Introduction

Set theory

Observation

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Kanamori (2008, 40):

> *"If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's forcing began its transformation into a modern, sophisticated one. [...] Set theory had undergone a sea-change."*

Gödel and the Continuum Hypothesis

By Gödel's Incompleteness Theorems (1931) there are always sentences that cannot be decided in a chosen axiomatization, i.e. they are independent from the chosen axiom system. *ZFC* is no exception.

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CH was shown to be *independent* from the standard axiomatization, i.e. it is possible to build models of

- $ZFC + CH$ (Gödel 1940);
- infinitely many versions of $ZFC + \neg CH$, according to how many cardinalities there are between the integers and real numbers (Cohen 1963, 1964).

The forcing method

Forcing is a technique that allows set theorists to build new set theoretic models "at will", according to their mathematical needs. In particular, forcing allows to build models that do or do not satisfy various (independent) sentences, i.e. $ZFC + A$ and $ZFC + \neg A$.

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Forcing is a powerful independence-proving technique!

... and much more!

Key notions of forcing

Forcing schema

We extend a model M of ZFC, the ground model, to a model $M[G]$ by adding a new object G that was not part of the ground model. This extension is a model of ZFC plus some additional statement that follows from G.

Forcing notion and generic filter

The new object G is a generic filter of a partial order $P = (P, \leq, 1), P \in M$, i.e. G meets every dense subset of P. Then $G \subset P, G \notin M, G \in M[G]$.

Forcing theorem

The *forcing language*: It contains a name for every element of $M[G]$, including a constant $\dot{G}$, the name for a generic set. Once a G is selected then every constant of the forcing language is interpreted as an element of the model $M[G]$.

The *forcing relation*: It is a relation between the forcing conditions and sentences of the forcing language: $p \Vdash \sigma$ (p *forces* σ); it is definable in M.

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Theorem

Let $(P, \leq)$ be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over M,

$$M[G] \models \sigma \quad \textit{if and only if} \quad (\exists p \in G) p \Vdash \sigma.$$

Impact of forcing results

In the presentation of the *Set-theoretic Pluralism Network* we read:

> *"Set theory is in the throes of a foundational crisis, the results of which may radically alter our understanding of the infinite and mathematics as a whole. In essence, the idea that there is a unique, so to speak, place in which all of mathematics occurs, has become increasingly controversial. There are a variety of reasons for this development, but a common thread among them is a growing acceptance of indeterminacy in the concept of set and in the foundations of mathematics more generally."*

Philosophy of set theory

Observation

Over the last years different programs emerged in the philosophy of set theory that are concerned with the changes in set theory that were introduced (among others) through forcing (Balaguer, Friedman et al, Hamkins, Shelah, Steel, Woodin, etc.).

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$\Rightarrow$ Universe/multiverse debate in the philosophy of set theory.

Universism

Universe View

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- There are definitive final answers to the question of whether a given mathematical statement, such as CH, is true or not, and set theorists seek to find these answers.
- The fact that such a statement is independent of ZFC or another weak theory is regarded as a distraction from the question of determining whether or not it is ultimately true.

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- Finding definitive final answers to the question of whether a given mathematical statement, such as CH, is true or not, is not possible or not desirable.
- There are diverse variations of multiversism.

The forcing technique and its results

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Philosophical upshots of forcing results: axiom candidates, multiverse picture, solutions to CH, etc.

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Philosophical upshots of forcing results: axiom candidates, multiverse picture, solutions to CH, etc.

Philosophical upshots of forcing technique: local arguments in universe/multiverse debate.

The results and technicalities of forcing

Example: "The set-theoretic multiverse" (Hamkins 2012) shows both lines of arguments:

1. The experience with most diverse models of set theory over the last decades implies a multiverse view.
2. The Naturalist account of forcing "implements in effect the content of the multiverse view".

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2. The Naturalist account of forcing "implements in effect the content of the multiverse view".

Aim

Investigate the method of forcing itself as one of the differentiating factors responsible for the philosophical conclusions that are drawn in recent programs in the philosophy of set theory.

Forcing technique in focus

Some meta-mathematics

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Traditionally, there are different approaches to defining forcing, via

1. the countable transitive model approach, or
2. Boolean-valued model approach.

The countable transitive model approach (CTMA)

CTMA in a nutshell

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Idea: There is an effective way of finding a finite set $\Lambda \subset ZFC$ such that we can prove in ZFC that if we have a ctm M of Λ then there is a generic extension $M[G]$ that satisfies some finite set $\Delta \subset ZFC$.

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Lemma

Let M be a ctm, P a forcing notion and $p \in P$. Then a generic filter G exists such that $p \in G$.

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Very rough outline:

- Start with a transitive model M of ZFC.
- Every partial order P (the forcing notion) can be embedded into a complete Boolean algebra B.
- The Boolean-valued model M^B: the elements correspond to the names of the forcing language; interpretation of formulas via assignment to elements of B (Boolean values).

Interrelations between forcing approaches

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Forcing approaches are meta-mathematically not equivalent.

We introduce the further claim that

Claim

Forcing approaches are not philosophically neutral.

In particular, the choice of forcing approach represents a *philosophical* rather than only a mathematical step *in a philosophical argument*.

The Hyperuniverse Program

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1. Use forcing and the inconsistencies it gives rise to as a tool to identify preferred models out of the current multiverse practice.
2. Extract generalized principles from these preferred models that lead to axioms which show directions in which *ZFC* can be extended.
3. No Platonistic background is assumed and these directions do not necessarily merge into one common extension. But in each individual direction, no inconsistencies remain (compartmentalization).

The Hyperuniverse

Definition (Hyperuniverse)

Let $\mathcal{H}^{ZFC}$ be the collection of all countable transitive models of ZFC. We call $\mathcal{H}^{ZFC}$ the hyperuniverse.

Friedman and Ternullo (2016, 176):

> *"[...] $\mathcal{H}$ is closed under forcing and inner models, which, as we saw, are the main techniques in the current practice. In other terms, if we start with countable transitive models, the use of forcing and inner models does not require more than and leave us with countable transitive models."*

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Choice to use CTMA (and therefore $\mathcal{H}^{ZFC}$ as the set-theoretic background) seems to be made out of mathematical expedience.

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1. Hamkins' toy model argument.
2. Relation to specific ontological conceptions.

Hamkins' toy model argument

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Conclusion: The toy model perspective (CTMA) can only serve "as a guide to the full, true, higher-order multiverse" (Hamkins 2012, 432).

CTMA and ontological conception

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Possible solutions:

1. Give an independent philosophical argument why this restrictiveness is desirable.
2. Use the forcing approach as an explication of the intended philosophical conception.

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- **Realism** about universes ("Platonistic multiverse"): Each set-theoretic universe is ontologically on a par with all others (even though some might still be preferred).
- The **truth** of *CH* is settled on the multiverse view by mathematicians' extensive knowledge about how it both holds and fails throughout the multiverse; it is incorrect to describe it as an open question.

The Multiverse Program and forcing approach

Hamkins (2012, 423):

> *"In any set-theoretic argument, a set theorist is operating in a particular universe V, conceived as the (current) universe of all sets, and whenever it is convenient he or she asserts 'let G be V-generic for the forcing notion $\mathcal{P}$,' and then proceeds to make an argument in $V[G]$, while retaining everything that was previously known about V and basic facts about how V sits inside $V[G]$."*

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Hamkins (2012), then, develops the Naturalist account of forcing that expresses the content of the multiverse view and that "seeks to legitimize the actual practice of forcing, as it is used by set theorists" (423).

Naturalist approach (NA)

NA in a nutshell

Similar to BVMA, but creates a two-valued class model expressing what it means to be a forcing extension via a Boolean ultrapower embedding.

Theorem (Naturalist account)

For any forcing notion P, there is an elementary embedding $V \preceq \bar{V} \subseteq \bar{V}[G]$ of the universe V into a class model $\bar{V}$ for which there is a $\bar{V}$-generic filter $G \subseteq \bar{P}$ and the entire extension $\bar{V}[G]$, including the embedding of V into $\bar{V}$, are definable classes in V and $G \in V$.

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So, the Naturalist account is not primarily interesting for mathematical reasons but for the possibility to argue mathematically for a philosophical point.

That is, the account explicates the philosophical idea of the program, the choice of the account is a philosophical step in this philosophical argument.

Forcing types

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Forcings that differ from set forcing in ZF(C):

1. Higher-order forcings: A-definable class forcing, class forcing in MK, hyperclass forcing in MK**, etc.
2. Forcing in theories weaker than ZF(C): ZFC^{-}, ill-founded models, etc.

Class forcing

Class forcing in a nutshell

The forcing notion P is class-sized instead of set-sized. For forcing to work correctly, some restrictions have to be put on P.

It can take the form of A-definable class forcing in a model of ZFC with an added class predicate A or general class forcing in MK.

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Example: The continuum function 2^κ can behave in any reasonable way for all regular cardinals κ (Easton forcing).

Class forcing and Hamkins multiverse

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Problems:

- Does it describe actual set-theoretic practice (as Hamkins claims)?
- Does it create a too restrictive multiverse (in contrast to Hamkins' aim)?
- Does it itself use a too restrictive notion of forcing (similar to toy model argument)?

Class forcing and NA

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Problem: The Boolean completion of a class partial order is a hyperclass; therefore BVMA and NA cannot be used to set up class forcing in ZFC (or even GBC).

Fact

For class forcing the CTMA and BVMA (or NA) are not mathematically equivalent.

Class forcing and NA

Possible solutions:

1. Give an external philosophical argument why the restriction to set forcing is warranted.
2. Change the set-theoretic explication to include class forcing.

Include class forcing?

Theorem (A., Friedman, Gitman)

In a model $V \models GBC$, a partial order P with a proper class antichain has a fully complete Boolean completion $\mathbf{B}_P$ if and only if $V \models MK$.

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Even more: To fully carry out the forcing construction via BVMA one needs to go to an extension of MK.

Possible solution: Set up a class multiverse, where the models are models of MK^+.

Conclusion and outlook

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2. Forcing is a multifaceted technique that allows for quite different (meta-)mathematical variations and philosophical uses.
3. The choices made in the technical forcing setup strongly inform (even determines) the philosophical results obtained. In particular, the forcing technique itself is not philosophically neutral.

Outlook

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Thank you...

...and questions, please!

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