G_δ sets in σ-ideals generated by compact sets

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- Going beyond that, we get "analytic sets" by projecting Borel sets
- Classical descriptive set theory: Borel heirarchy, projective heirarchy

For this talk:

- Open, closed, G_δ
- Analytic: projections of Borel sets
- Coanalytic

The hyperspace of compact sets

E a compact metric space.

$$\mathcal{K}(E) = \{F \subseteq E : F \text{ compact}\}$$

Hausdorff metric d_H on $\mathcal{K}(E)$ satisfies, for nonempty $F, K \in \mathcal{K}(E)$,

$$d_H(F,K) < \epsilon \iff F \subseteq B(K,\epsilon) \text{ and } K \subseteq B(F,\epsilon)$$

where $B(K,\epsilon) = \bigcup_{x \in K} B(x,\epsilon)$.

This makes $\mathcal{K}(E)$ into a compact metric space.

Definitions: ideals and σ-ideals of compact sets

A set $I \subseteq \mathcal{K}(E)$ is an *ideal of compact sets* if it is:

1. downward closed: if $F_1 \subseteq F_2$ and $F_2 \in I$ then $F_1 \in I$
2. closed under finite union: if $F_1, \ldots, F_k \in I$ then $\bigcup_1^k F_i \in I$.

An ideal I is a *σ-ideal of compact sets* if it is also closed under countable unions whenever the union itself is compact.

Examples

Ideals of compact sets arise out of various notions of smallness. Examples include ideals of the following compact sets:

- meager sets
- null sets for a finite Borel measure
- sets of dimension $\leq n$ for fixed $n \in \mathbb{N}$
- zero sets with respect to a Hausdorff measure
- $\mathcal{Z}$-sets for $E = [0,1]^\omega$

(In all these examples we consider compact sets only)

Examples

- the compact subsets of $\mathbb{Q}$
- the countable compact subsets of $\mathbb{R}$
- compact sets of uniqueness:
  A set of uniqueness is a subset of the unit circle for which every trigonometric series $\Sigma c_n e^{inx}$ converging to 0 outside the set is identically 0.

Complexity of an ideal

The condition of being an ideal or σ-ideal is strongly related to the complexity of $I \subseteq \mathcal{K}(E)$.

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- Kechris–Louveau, Dougherty: if I is a G_δ ideal, it must be a σ-ideal
- Kechris–Louveau–Woodin: if a σ-ideal I is either co-analytic or analytic, it must be either complete co-analytic or simply G_δ.

Property $(*)$

We consider G_δ σ-ideals of compact sets that satisfy the following natural condition, formulated by Solecki:

$I \subseteq K(E)$ has *property* $(*)$ if, for any sequence of sets $K_n \in I$, there exists a G_δ set G such that $\bigcup_n K_n \subseteq G$ and $\mathcal{K}(G) \subseteq I$.

Comments on property $(*)$

If I has $(*)$ and is analytic or co-analytic, it must be a G_δ σ-ideal (Solecki)

Question: Does every G_δ ideal have property $(*)$? No, but all the natural examples of G_δ ideals have it.

A representation theorem

For $A \subseteq E$, define

$$A^* = \{K \in \mathcal{K}(E) : K \cap A \neq \emptyset\}.$$

Theorem (Solecki). Suppose I is co-analytic and non-empty. Then I has property $(*)$ iff there exists a closed set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that, for any $K \in \mathcal{K}(E)$,

$$K \in I \iff K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

Easiest example: Lebesgue measure

For E the unit square in the plane and λ Lebesgue measure on E, let

$$I = \{K \in \mathcal{K}(E) : \lambda(K) = 0\}$$

Then the set

$$\mathcal{F} = \{K \in \mathcal{K}(E) : \lambda(K) \geq 1/2\}$$

works to characterize membership in the ideal.

Note that given an ideal I, the representing set $\mathcal{F}$ is not unique.

What about G_δ subsets of E?

Given a set $\mathcal{F} \subseteq \mathcal{K}(E)$, we may consider G_δ sets $G \subseteq E$ for which G^* is meager in $\mathcal{F}$.

Question:
For which G_δ sets $G \subseteq E$ will we have G^* meager in $\mathcal{F}$?

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In other words: For which G will the test upstairs identify G as a "small" set downstairs?

What do we get in the case of G_δ subsets of E?

For example, in the case of measure, suppose we have $\mathcal{F}$ that works to identify compact sets of measure zero. We can apply the same test (meagerness of G^* in $\mathcal{F}$) to G_δ sets G. Which G_δ sets will end up having G^* meager in $\mathcal{F}$?

Theorem

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Let I be a G_δ ideal of compact sets, with property $(*)$. Then there exists a nonempty sets $\mathcal{F} \subseteq \mathcal{K}(E)$, such that for any G_δ subset G of E,

$$G^* \text{ meager in } \mathcal{F} \iff G \subseteq \bigcup_{n\in\mathbb{N}} K_n \text{ for some } K_n \in I.$$

Note that for a closed set K this condition implies that K^* is meager in $\mathcal{F}$ if and only if K is in I.

Proof

Take two disjoint open sets U, V

Construct a set $\mathcal{F}$ such that, for G_δ sets $G \subseteq U$,
G^* meager in $\mathcal{F} \iff G \subseteq \bigcup_{n\in\mathbb{N}} K_n$ for some $K_n \in I$.

Put some "code sets" inside of V

$F \in \mathcal{F}$ if the complement of F in U is covered in a particular way:

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If $G \subseteq U$ is a G_δ set not covered by countably many sets in I then we play the Banach-Mazur game to show that G^* is comeager in some open subset of $\mathcal{F}$.

References

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