Portrait of the intuitionist as an Opponent in Dialogues.

Embedding brouwerian counter-examples within classical plays

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Brouwer (1881-1966) explicitly rejected axiomatic systems and he has never used any intuitionistic logical framework to express his mathematical results. Intuitionistic logic is due to Heyting. According to Brouwer, mathematics is not a science, but it consists in acts.

His counter-examples to the excluded-middle and to the accorded view on the continuum are such acts.
Some of them are based on a device, which has been later axiomatized by G. Kreisel: the *Creative Subject*.

$$\forall n \Sigma \vdash_n A \vee \neg \Sigma \vdash_n A (CS1)$$

$$\Sigma \vdash_m A \rightarrow \ \Sigma \vdash_{m+n} A (CS2)$$

$$A \leftrightarrow \exists n \Sigma \vdash_n A (CS3)$$

Nevertheless (Sundholm 2015) has convincingly argued that these axioms are classicaly valid.
According to him, the creative subject, as rendered through Kreisel's axioms, is a dead-end.
But it does not mean that Brouwer's view as such is wrong.
Can we make sense of his counter-arguments without basing them on axioms, but on his own view on mathematics?

Wheresoever in logic the word "all" and "every" is used, this word, in order to make sense, tacitly involves the restriction: insofar as belonging to a mathematical structure which is supposed to be constructed beforehand. (Brouwer 1907)

Human understanding is based upon the construction of common mathematical systems, in such a way that for each individual an element of life is connected with the same element of such a system. (Brouwer 1907)

1. Standard Dialogical Logic

2. Local reasons and material dialogues

3. Dialogical reconstruction of some of Brouwer's counter-examples

1. Standard Dialogical Logic

Conjonction	X! $A \wedge B$	Y? A Y? B	X! A X! B
Disjonction	X! $A \vee B$	Y?	X! A X! B
Implication	X! $A \rightarrow B$	Y! A	X! B
Negation	X! $\neg A$	Y! A	$\varnothing$
Universal	X! $\forall x P(x)$	Y? n	$P(n)$
Existential	X! $\exists x P(x)$	Y?	$P(n)$

Table: particle rules

1. Standard Dialogical Logic

Structural Rules:

- Dialogues about propositions consist of *arguments* which are put forth *alternatively* by an opponent **O** and a proponent **P**. The arguments follow certain *rules of argumentation* that belong to the game such that each play ends up with *win* or *loss* for either player.
- Each argument either attacks prior ones or defends those of one's own upon such an attack[1]
- Whoever cannot put forth an argument any longer has *lost* that play of the game; the other one has *won* it.

[1] there is an additional *repetition rule*, which is intended to avoid infinite plays, based on an indefinite repetition of the same attack, see (Lorenz 2010)

1. Standard Dialogical Logic

Structural Rules:

P cannot play an elementary statement if **O** has not stated it previously. (copy-cat rule)

1. Standard Dialogical Logic

O			P		
				$! \ \forall x(A(x) \vee \neg A(x))$	0
1	$?^a$	0		$!A(a) \vee \neg A(a)$	2
3	$?^\vee$	2		$! \ \neg A(a)$	4
5	$!A(a)$		4	-	
3'	$(?^\vee)$	0		$!A(a)$	6

Table: The excluded-middle, *classical* play

2. Local reasons and material dialogues

In (Rahman, Clerbout, 2015) and in (Rahman et al., 2018), the Standard Dialogic has been linked to Martin-Löf's Constructive Type Theory (Martin-Löf, 1984).
CTT is based on judgments, i.e. propositions associated with an assertive force which is backed up by *proof-objects*.
A proof-object is an object whose construction holds as a proof of a proposition (which can be seen as the set of all its proofs, due to the Curry-Howard correspondence). The notation is :

$$a : A$$

2. Local reasons and material dialogues

By distinguishing the *play-level* and the *strategic level*, the dialogical logic introduces a distinction that we do not have in the CTT.

The play level corresponds to particular plays, in which the players play according to the rules.

The strategy standpoint is a generalization of the procedure which is implemented at the play level; it is a systematic exposition of all the relevant variants of a game.

A *strategic reason* is something whose possession enables a player to win whatever is the strategy of the other player. It corresponds to a proof-object in the CTT.

A *local reason* is a justification, supplied within a particular play by a player, in order to back up a statement. It corresponds to nothing in the CTT.

2. Local reasons and material dialogues

When a *local reason* is adduced within a dialogue, in defence of a proposition, it must be though of as prefiguring a material dialogue, whose rules are specific to this very proposition (cf. Wittgenstein's *Language games*).

2. Local reasons and material dialogues

O			P		
				! $(\forall x : E)P(x) \vee \neg P(x)$	0
1	$m = 1$			$n = 2$	2
3	$e_1 : E$	0		$e_2 : P(e_1) \vee \neg P(e_1)$	4
5	$?^{\vee}$	4		$R^{\vee}(e_2) : \neg P(e_1)$	6
7	$?^{\cdots}/R^{\vee}(e_2)$	6		$f : \neg P(e_1)$	8
9	$L^{\rightarrow}(f) : P(e_1)$	8			
11	$g : P(e_1)$			$?^{\cdots}/L^{\rightarrow}(f)$	10
5'	$(?^{\vee})$	4		$g : P(e_1)$	12
13	$g =?$	12		$g = L^{\rightarrow}(f) : P(e_1)$	14

Table: Dialogue on the Excluded-middle, with local reasons

2. Local reasons and material dialogues

potentiality and actuality of constructions

In order to make sense of the way Brouwer proceeds in his counter-examples, we shall additionally make sense of the distinction between *potentiality* and *actuality* for local reasons, by distinguishing:

$\langle\ \rangle_a : A$: local reason or *purely potential* construction of a sequence developing it.

$\langle 0, ..., n\rangle_a : A$: actual finite sequence developing the local reason a

$\langle 0, ..., n, [i]^X\rangle_a : A$ or $\langle [i]^X\rangle_a : A$: actual finite sequence developing the local reason a and in which a certain part (respectively all of it), including intensional compounds, remains private to the player X.

2. Local reasons and material dialogues

potentiality and actuality of constructions

R1. If **O** define an entity, he starts an empty sequence.
R2. When **O** starts an empty sequence, he decides whether it will correspond to a *lawlike box* or not. If it is not the case, the empty sequence is marked by the sign ∞ and **P** is considered as not being able to guess the rule according to which the values of the sequence are being chosen.
R3. If the sequence is not lawlike, **O** is allowed to put the sequence he is constructing and the involved instructions in the box. **P** can ask to open the box when he is asked to resolve instructions, by applying it to a determinate content that may be in the box. If the box is not such as he assumed, he has lost the play.

3. reconstruction of Brouwer's counter-examples

from *Mathematics, Science and Language*(1929)

A *fleeing property* is defined as a property for which in the case of each natural number one can prove either that it exists or that it is absurd, while one cannot calculate a particular number that has the property, nor one can prove the absurdity of the property for all natural numbers. The *critical number* λ_f of a fleeing property f is defined as the (hypothetical) smallest natural number that possesses the property. An *up number* and a *down number* of f is a number that is, respectively, not smaller and smaller than the critical number. Obviously, each natural number can be recognized to be either an up number or a down number. In the first case, the property loses its character of being *fleeing*. According to these conditions, we can now define the *binary oscillatory number* p_f as the real number determined by the limit of the convergent sequence $a_1, a_2, \ldots$, where a_ν for an arbitrary **down** number ν of f is equal to $(-\frac{1}{2})^\nu$ and for an arbitrary **up** number ν of f is equal to $(-\frac{1}{2})^{\lambda_f}$. This binary oscillatory number is neither equal to 0 nor distinct from 0, which contradicts the Principle of the Excluded Middle.

3. reconstruction of Brouwer's counter-examples

from *Mathematics, Science and Language*(1929), dialogical reconstruction

O			P		
				$(\forall x \in \mathbb{R})(x = 0 \vee \neg(x = 0))$	0
1	$\langle\,\rangle_{p_f} : p_f \in \mathbb{R}$	0		$!\ p_f = 0 \vee \neg(p_f = 0)$	2
3	$?^{\vee}$	2		$L^{\vee}(d_{p_f}) : \neg(p_f = 0)$	4
5	$\cdots / L^{\vee}(d_{p_f})$	4		$c(\lambda_f) : \neg(p_f = 0)$	6
7	$?c(\lambda_f)$	6		-	
3'	$(?^{\vee})$			-	

Table: dialogical reconstruction of the first counter-example. **O wins**

3. reconstruction of Brouwer's counter-examples

from *Mathematics, Science and Language*(1929), alternate dialogical reconstruction

O			P		
				$(\forall x \in \mathbb{R})(x = 0 \vee \neg(x = 0))$	0
1	$\langle\ \rangle_{p_f} : p_f \in \mathbb{R}$	0		$!\ p_f = 0 \vee \neg(p_f = 0)$	2
3	$?^{\vee}$	2		$L^{\vee}(d_{p_f}) : \neg(p_f = 0)$	4
5	$\cdots / L^{\vee}(d_{p_f})$	4		$c(\lambda_f) : \neg(p_f = 0)$	6
7	$L^{\rightarrow}(c(\lambda_f)) : p_f = 0$				
9	$!p_f = 0$		7	$\cdots / L^{\rightarrow}c(\lambda_f)$	8
3'	$(?^{\vee})$			$!p_f = 0$	10
11	$?!$	10			
13	$d(c(\langle 0, \ldots, \llcorner\lambda_f^{?}\lrcorner, \ldots\rangle))$		9	$?!$	12

Table: alternate dialogical reconstruction of the first counter-example. **O wins**

3. reconstruction of Brouwer's counter-examples

from *The Principles of Continuum*(1930)

Let p be an element determined by the convergent sequence $c_1, c_2...$, for which I choose c_1 to be the zero point and every $c_{\nu+1} = c_\nu$ with one exception: As soon as I find a critical number λ_f of a certain fleeing property f, I choose the next c_ν to be equal to $-2^{-\nu-1}$ and as soon as I find a proof of the absurdity of such a critical number, I choose the next c_ν to be equal to $2^{-\nu-1}$. This element is distinct from zero, and yet it is neither smaller nor greater than zero.

3. reconstruction of Brouwer's counter-examples

from *The Principles of Continuum*(1930), dialogical reconstruction

O			P		
				$! \ \forall x \in \mathbb{R}(x \neq 0 \rightarrow (x < 0 \vee x > 0))$	0
1	$\langle \ \rangle_p^{\infty} : p \in \mathbb{R}$			$p \neq 0 \rightarrow (p < 0 \vee p > 0)$	2
3	$\langle [i]^{\mathbf{O}} \rangle_p : p \neq 0$	2		$b(\langle [i]^{\mathbf{O}} \rangle_p) : p < 0 \vee p > 0$	4
5	$?^{\vee}$	4		$L^{\vee}(b(\langle [i]^{\mathbf{O}}_{(?\lambda_f)} \rangle_p)) : p < 0$	6
7	$?^{\cdots}/L^{\vee}(b(\langle [i]^{\mathbf{O}}_{(?\lambda_f)} \rangle_p))$	6			
9	$\lambda_f \notin \langle [i]^{\mathbf{O}} \rangle_p$			**Open** $\langle [i]^{\mathbf{O}}_n \rangle_p$	8
5'	$(?^{\vee})$	4		$L^{\vee}(b(\langle 0, \ldots, n, [i]^{\mathbf{O}}_{(?e:\lambda_f \rightarrow \bot)} \rangle_p)) : p > 0$	10
11	$?^{\cdots}/L^{\vee}(b(\langle 0, \ldots, n, [i]^{\mathbf{O}}_{(?e:\lambda_f \rightarrow \bot)} \rangle_p))$	10			
13	$e \notin \langle 0, \ldots, n, [i]^{\mathbf{O}} \rangle_p$			**Open** $\langle 0, \ldots, n, [i]^{\mathbf{O}}_{n+k} \rangle_p$	12

Table: dialogical reconstruction of the second counter-example

Thank you!