Logics for Rough Concept Analysis

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Motivation and Aim

- *Formal contexts*, or *polarities*, are structures $\mathbb{P} = (A, X, I)$ such that A and X are sets, and $I \subseteq A \times X$ is a binary relation. Intuitively, formal contexts can be understood as abstract representations of databases.
  For any relation $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, let

$$T^{(0)}[V'] := \{u \mid \forall v(v \in V' \Rightarrow uTv)\} \qquad T^{(1)}[U'] := \{v \mid \forall u(u \in U' \Rightarrow uTv)\}.$$

Motivation and Aim

Definition

For every formal context $\mathbb{P} = (A, X, I)$, a *formal concept* of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^{\uparrow} = Y$ and $Y^{\downarrow} = B$. The set B is the *extension* of c, denoted by $[\![c]\!]$, and Y is the *intension* of c, denoted $(\![c]\!)$.

Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the *concept lattice* of $\mathbb{P}$ is the complete lattice $\mathbb{P}^+ := (L(\mathbb{P}), \bigwedge, \bigvee)$,where for every $\mathcal{X} \subseteq L(\mathbb{P})$,

$$\bigwedge \mathcal{X} := (\bigcap_{c \in \mathcal{X}} [\![c]\!], (\bigcap_{c \in \mathcal{X}} [\![c]\!])^{\uparrow}) \quad \text{and} \quad \bigvee \mathcal{X} := ((\bigcap_{c \in \mathcal{X}} (\![c]\!))^{\downarrow}, \bigcap_{c \in \mathcal{X}} (\![c]\!)).$$

- $\top^{\mathbb{P}^+} := \bigwedge \varnothing = (A, A^{\uparrow})$ and $\bot^{\mathbb{P}^+} := \bigvee \varnothing = (X^{\downarrow}, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad [\![c]\!] \subseteq [\![d]\!] \quad \text{iff} \quad (\![d]\!) \subseteq (\![c]\!).$$

Motivation and Aim

Definition

An *enriched formal context* is a tuple $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$ such that $\mathbb{P} = (A, X, I)$ is a formal context, and $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$ are *I-compatible* relations, that is, $R_\Box^{(0)}[x]$ (resp. $R_\Diamond^{(0)}[a]$) and $R_\Box^{(1)}[a]$ (resp. $R_\Diamond^{(1)}[x]$) are Galois-stable for all $x \in X$ and $a \in A$. The *complex algebra* of $\mathbb{F}$ is

$$\mathbb{F}^+ = (\mathbb{P}^+, [R_\Box], \langle R_\Diamond \rangle),$$

where $\mathbb{P}^+$ is the concept lattice of $\mathbb{P}$, and

$$[R_\Box]c := (R_\Box^{(0)}[([c])], (R_\Box^{(0)}[([c])])^\uparrow)$$

and $$\langle R_\Diamond \rangle c := ((R_\Diamond^{(0)}[[\![c]\!]])^\downarrow, R_\Diamond^{(0)}[[\![c]\!]]).$$

Motivation and Aim

Definition

Given I-compatible relations $R, T \subseteq A \times X$, the *composition* $R\,;T \subseteq A \times X$ is defined as:

$$(R\,;T)^{(1)}[a] = R^{(1)}[I^{(0)}[T^{(1)}[a]]] \textit{ or equivalently } (R\,;T)^{(0)}[x] = R^{(0)}[I^{(1)}[T^{(0)}[x]]].$$

Rough formal contexts are tuples $\mathbb{G} = (\mathbb{P}, E)$ such that $\mathbb{P} = (A, X, I)$ is a polarity, and $E \subseteq A \times A$ is an equivalence relation. For every $a \in A$ we let $(a)_E := \{b \in A \mid aEb\}$. The relation E induces two relations $R, S \subseteq A \times I$ approximating I, defined as follows:

$$aRx \text{ iff } bIx \text{ for some } b \in (a)_E; \qquad aSx \text{ iff } bIx \text{ for all } b \in (a)_E. \tag{1}$$

Motivation and Aim

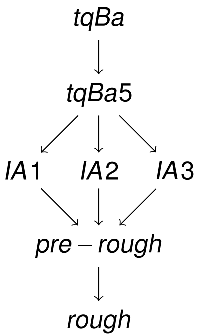
Definition

$\mathbb{T} = (\mathbb{L}, I)$ is a *topological quasi-Boolean algebra*(tqBa) if $\mathbb{L} = (L, \vee, \wedge, \neg, \top, \bot)$ is a De Morgan algebra and for all $a, b \in L$,

T1. $I(a \wedge b) = Ia \wedge Ib$, T2. $IIa = Ia$, T3. $Ia \leq a$, T4. $I\top = \top$.

Algebras	Acronyms	Axioms
topological quasi Boolean algebra 5	tqBa5	T5: $CIa = Ia$
intermediate algebra of type 1	IA1	T5, T6: $Ia \vee \neg Ia = \top$
intermediate algebra of type 2	IA2	T5, T7: $Ia \vee Ib = I(a \vee b)$
intermediate algebra of type 3	IA3	T5, T8: $Ia \leq Ib$ and $Ca \leq Cb$ imply $a \leq b$
pre-rough algebra	pra	T5, T6, T7, T8.

Motivation and Aim



Motivation and Aim

Kent's rough formal concepts

Consider a formal context $\mathbb{F} = (G, M, I)$ with an approximation space (G, E) on objects. We define upper and lower approximations for $\mathbb{F}$ as $(G, M, \bar{I})$ and $(G, M, \underline{I})$ respectively defined as

1. $g\bar{I}m$ iff there exists $m' \in M$ such that, mEm' and gIm
2. $g\underline{I}m$ iff for all $m' \in M$, mEm' implies gIm

Upper and lower approximations for concept (A, B) are defined as $(\bar{I}^{(0)}(B), I^{(1)}(\bar{I}^{(0)}(B)))$ and $(\underline{I}^{(0)}(B), I^{(1)}(\underline{I}^{(0)}(B)))$ respectively.

Introduction

Definition

An Rfc $\mathbb{G} = (\mathbb{P}, E)$ is *amenable* if E, R and S are I-compatible.

- R - lax operators - $\Box_\ell, \Diamond_\ell$
- S - strict operators - $\Box_s, \Diamond_s$
- For any amenable Rfc,

$$\Box_s\phi \vdash \phi \quad \phi \vdash \Box_\ell\phi \quad \phi \vdash \Diamond_s\phi \quad \Diamond_\ell\phi \vdash \phi, \tag{2}$$

Lemma

For any amenable Rfc $\mathbb{G} = (\mathbb{P}, E)$, if and R and S are defined as in (1), *then*

$$R; R \subseteq R \quad \text{and} \quad S \subseteq S; S(3)$$

Introduction

$$\Box_s\phi \vdash \Box_s\Box_s\phi \quad \Box_\ell\Box_\ell\phi \vdash \Box_\ell\phi \quad \Diamond_s\Diamond_s\phi \vdash \Diamond_s\phi \quad \Diamond_\ell\phi \vdash \Diamond_\ell\Diamond_\ell\phi (4)$$

$$\phi \vdash \Box_s\Diamond_s\phi \qquad \Diamond_s\Box_s\phi \vdash \phi \qquad \phi \vdash \Box_\ell\Diamond_\ell\phi \qquad \Diamond_\ell\Box_\ell\phi \vdash \phi (5)$$

We define Kent algebras as motivated from above:

Definition

A *basic Kent algebra* is a structure $\mathbb{A} = (\mathbb{L}, \Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell)$ such that $\mathbb{L}$ is a complete lattice, and $\Box_s, \Diamond_s, \Box_\ell, \Diamond_\ell$ are unary operations on $\mathbb{L}$ such that for all $a, b \in \mathbb{L}$,

$$\Diamond_s a \leq b \text{ iff } a \leq \Box_s b \quad \text{and} \quad \Diamond_\ell a \leq b \text{ iff } a \leq \Box_\ell b, \tag{6}$$

$$\Box_s a \leq a \qquad a \leq \Diamond_s a \qquad a \leq \Box_\ell a \qquad \Diamond_\ell a \leq a \tag{7}$$

$$\Box_s a \leq \Box_s\Box_s a \qquad \Diamond_s\Diamond_s a \leq \Diamond_s a \qquad \Box_\ell\Box_\ell a \leq \Box_\ell a \qquad \Diamond_\ell a \leq \Diamond_\ell\Diamond_\ell a \tag{8}$$

Introduction

Definition

An aKa $\mathbb{A}$ is an *aKa5'* if for any $a \in \mathbb{L}$,

$$\Diamond_\ell a \leq \Box_s \Diamond_\ell a \qquad \Diamond_s \Box_\ell a \leq \Box_\ell a \qquad \Box_s a \leq \Diamond_\ell \Box_s a \qquad \Box_\ell \Diamond_s a \leq \Diamond_s a; \quad (9)$$

is a *K-IA3$_s$* if for any $a, b \in \mathbb{L}$,

$$\Box_s a \leq \Box_s b \text{ and } \Diamond_s a \leq \Diamond_s b \text{ imply } a \leq b, \quad (10)$$

and is a *K-IA3$_\ell$* if for any $a, b \in \mathbb{L}$,

$$\Box_\ell a \leq \Box_\ell b \text{ and } \Diamond_\ell a \leq \Diamond_\ell b \text{ imply } a \leq b. \quad (11)$$

Multi-type environment

Decompositions of unary operators

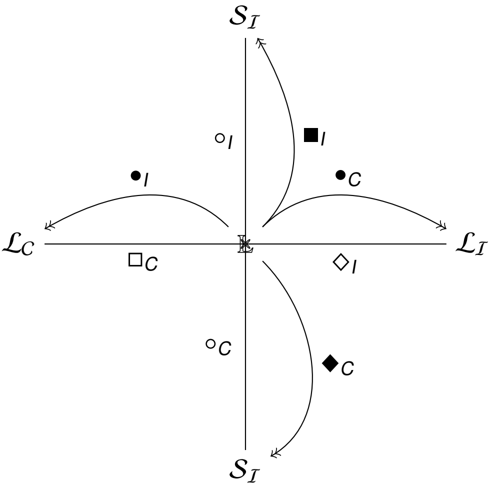
$\Box_s = \circ_i \cdot \blacksquare_i$	$\blacksquare_i \cdot \circ_i = id_{S_I}$	$\Diamond_s = \circ_c \cdot \blacklozenge_c$	$\blacklozenge_c \cdot \circ_c = id_{S_C}$
$\circ_i : S_I \hookrightarrow \mathbb{L}$	$\blacksquare_i : \mathbb{L} \twoheadrightarrow S_I$	$\blacklozenge_c : \mathbb{L} \twoheadrightarrow S_C$	$\circ_c : S_C \hookrightarrow \mathbb{L}$

$\Box_\ell = \Box_c \cdot_c c$	$\bullet_C \cdot \Box_c = id_{L_C}$	$\Diamond_\ell = \Diamond_i \cdot \bullet_I$	$\bullet_I \cdot \Diamond_i = id_{L_I}$
$\bullet_C : \mathbb{L} \twoheadrightarrow L_C$	$\Box_c : L_C \hookrightarrow \mathbb{L}$	$\Diamond_i : L_I \hookrightarrow \mathbb{L}$	$\bullet_I : \mathbb{L} \twoheadrightarrow L_I$

where $S_I := \Box_s[\mathbb{L}]$, $S_C := \Diamond_s[\mathbb{L}]$, $L_C := \Box_\ell[\mathbb{L}]$, and $L_I := \Diamond_s[\mathbb{L}]$, and such that for all $\alpha \in S_I$, $\delta \in S_C$, $a \in \mathbb{L}$, $\pi \in L_I$, $\sigma \in L_C$,

$$\circ_i\alpha \leq a \text{ iff } \alpha \leq \blacksquare_i a \quad \blacklozenge_c a \leq \delta \text{ iff } a \leq \circ_c\delta \quad \bullet_C\, a \leq \pi \text{ iff } a \leq \Box_c\pi \quad \Diamond_i\sigma \leq a \text{ iff } \sigma \leq \bullet_i a. \tag{12}$$

Multi-type environment



Multi type environment

Definition

For any aKa $\mathbb{A}$, the *strict interior kernel* $S_I = (S_I, \cup_I, \cap_I, t_I, f_I)$ and the *strict closure kernel* $S_C = (S_C, \cup_C, \cap_C, t_C, f_C)$ are such that, for all $\alpha, \beta \in S_I$, and all $\delta, \gamma \in S_C$,

$$\alpha \cup_I \beta := \blacksquare_i(\circ_i \alpha \vee \circ_i \beta) \qquad \delta \cup_C \gamma := \blacklozenge_c(\circ_c \delta \vee \circ_c \gamma)$$
$$\alpha \cap_I \beta := \blacksquare_i(\circ_i \alpha \wedge \circ_i \beta) \qquad \delta \cap_C \gamma := \blacklozenge_c(\circ_c \delta \wedge \circ_c \gamma)$$
$$\top_i := \blacksquare_i \top, \ \bot_i := \blacksquare_i \bot \qquad \top_c := \blacklozenge_c \top, \ \bot_c = \blacklozenge_c \bot$$

Heterogeneous Algebras

Definition

A *heterogeneous aKa* (haKa) is a tuple

$$\mathbb{H} = (\mathbb{L}, S_I, S_C, L_I, L_C, \circ_i, \blacksquare_i, \circ_c, \blacklozenge_c, \bullet_I, \Diamond_i, \bullet_C, \Box_c)$$

such that:

H1 $\mathbb{L}, S_I, S_C, L_I, L_C$ are bounded lattices;

H2 $\circ_i : S_I \hookrightarrow \mathbb{L}$, $\circ_c : S_C \hookrightarrow \mathbb{L}$, $\bullet_I : \mathbb{L} \twoheadrightarrow L_I$, $\bullet_C : \mathbb{L} \twoheadrightarrow L_C$ are lattice homomorphisms;

H3 $\circ_i \dashv \blacksquare_i \qquad \blacklozenge_c \dashv \circ_C \qquad \bullet_C \dashv \Box_c \qquad \Diamond_i \dashv \bullet_i$;

H4 $\blacksquare_i \circ_i = id_{S_I} \qquad \blacklozenge_c \circ_C = id_{S_C} \qquad \bullet_C \Box_c = id_{L_C} \qquad \bullet_i \Diamond_i = id_{L_I}$ [1]

[1] Condition H3 implies that $\blacksquare_i : \mathbb{L} \twoheadrightarrow S_I$ and $\Box_i : L_I \hookrightarrow \mathbb{L}$ are $\wedge$-hemimorphisms and $\blacklozenge_C : \mathbb{L} \twoheadrightarrow S_C$ and $\Diamond_C : L_C \hookrightarrow \mathbb{L}$ are $\vee$-hemimorphisms; condition H4 implies that the black connectives are surjective and the white ones are injective.

Heterogeneous Algebras

The haKas corresponding to the varieties of Definition 8 are defined as follows:

Algebra	Acronym	Conditions
heterogeneous aKa5'	haKa5'	$\Diamond_i\pi \leq \circ_i\blacksquare_i\Diamond_i\pi$ $\quad \circ_c\blacklozenge_c\Box_c\sigma \leq \Box_c\sigma$ $\circ_i\alpha \leq \Diamond_i\bullet_I\circ_i\alpha$ $\quad \Box_c\bullet_C\circ_c\delta \leq \circ_c\delta$
heterogeneous K-IA3$_s$	hK-IA3$_s$	$\blacksquare_i a \leq \blacksquare_i b$ and $\blacklozenge_c a \leq \blacklozenge_c b$ imply $a \leq b$
heterogeneous K-IA3$_\ell$	hK-IA3$_\ell$	$\Box_C \bullet a \leq \Box_c \bullet b$ and $\Diamond_I \bullet a \leq \Diamond_I \bullet b$ imply $a \leq b$

Notice that the inequalities defining haKa5' are all analytic inductive. Defining rule for hK-IA3$_\ell$ is converted in multitype environment to

$$a \wedge \Box_I \bullet_I b \leq \Diamond_C \bullet_C a \vee b.$$

which is also analytic inductive however corresponding inequality for hK-IA3$_s$ is not analytic inductive.

Heterogeneous Algebras

Theorem

For every $\mathbb{K} \in \{aKa, aKa5', K\text{-}IA3_s, K\text{-}IA3_\ell\}$, letting $\mathbb{HK}$ denote its corresponding class of heterogeneous algebras, the following holds:

1. *If $\mathbb{A} \in \mathbb{K}$, then $\mathbb{A}^+ \in \mathbb{HK}$;*
2. *If $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}_+ \in \mathbb{K}$;*
3. $\mathbb{A} \cong (\mathbb{A}^+)_+$ *and* $\mathbb{H} \cong (\mathbb{H}_+)^+$.
4. *The isomorphisms of the previous item restrict to perfect members of $\mathbb{K}$ and $\mathbb{HK}$.*
5. *If $\mathbb{A} \in \mathbb{K}$, then $\mathbb{A}^\delta \cong ((\mathbb{A}^+)^\delta)_+$ and if $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}^\delta \cong ((\mathbb{H}_+)^\delta)^+$.*

Multi-type calculi

Language

general lattice L

$$A ::= p \mid \top \mid \bot \mid \circ_I \alpha \mid \circ_C \delta \mid \Diamond_I \pi \mid \Box_C \sigma \mid A \wedge A \mid A \vee A$$
$$X ::= A \mid \check{\bot} \mid \hat{\top} \mid \tilde{\circ}_I \Gamma \mid \tilde{\circ}_C \Delta \mid \hat{\Diamond}_I \Pi \mid \check{\Box}_I \Pi \mid \hat{\Diamond}_C \Sigma \mid \check{\Box}_C \Sigma \mid X \hat{\wedge} X \mid X \check{\vee} X$$

strict-interior kernel S_I

$$\alpha ::= \blacklozenge_I A \mid \blacksquare_I A$$
$$\Gamma ::= \alpha \mid \hat{\blacklozenge}_I X \mid \check{\blacksquare}_I X \mid \check{f}_I \mid \hat{t}_I \mid \Gamma \hat{\cap}_I \Gamma \mid \Gamma \check{\cup}_I \Gamma$$

lax-interior kernel L_I

$$\pi ::= \bullet_I A$$
$$\Pi ::= \pi \mid \tilde{\bullet}_I X \mid \check{0}_I \mid \hat{1}_I \mid \Pi \hat{\sqcap}_I \Pi \mid \Pi \check{\sqcup}_I \Pi$$

strict-closure kernel S_C

$$\delta ::= \blacklozenge_C A \mid \blacksquare_C A$$
$$\Delta ::= \delta \mid \hat{\blacklozenge}_C X \mid \check{\blacksquare}_C X \mid \check{f}_C \mid \hat{t}_C \mid \Delta \hat{\cap}_C \Delta \mid \Delta \check{\cup}_C \Delta$$

lax-closure kernel L_C

$$\sigma ::= \bullet_C A$$
$$\Sigma ::= \sigma \mid \tilde{\bullet}_C X \mid \check{0}_C \mid \hat{1}_C \mid \Sigma \hat{\sqcap}_C \Sigma \mid \Sigma \check{\sqcup}_C \Sigma$$

Multi-type calculi

The calculus $D.AKA$ consists of the following axiom and rules.

- Identity and Cut:

$$Id_L \; \frac{}{p \vdash p} \qquad \frac{x \vdash a \qquad a \vdash y}{x \vdash y} \; \mathrm{Cut}$$

- Multi-type display rules:

$$ad_{LS_I} \; \frac{\tilde{\circ}_I \Gamma \vdash X}{\Gamma \vdash \check{\blacksquare}_I X} \qquad \frac{X \vdash \tilde{\circ}_I \Gamma}{\hat{\blacklozenge}_I X \vdash \Gamma} \; ad_{LS_I}$$

$$ad_{LS_C} \; \frac{X \vdash \tilde{\circ}_C \Delta}{\hat{\blacklozenge}_C X \vdash \Delta} \qquad \frac{\tilde{\circ}_C X \vdash \Delta}{X \vdash \check{\blacksquare}_C \Delta} \; ad_{LS_C}$$

Multi-type calculi

- Multi-type structural rules for strict-kernel operators:

$$\tilde{\circ}\,\hat{t}_I\ \frac{\tilde{\circ}_I\,\hat{t}_I \vdash X}{\hat{\top} \vdash X} \qquad \frac{X \vdash \tilde{\circ}_I\,\check{t}_I}{X \vdash \check{\bot}}\ \tilde{\circ}_I\,\check{t}_I \qquad \tilde{\circ}_C\,\hat{t}_C\ \frac{\tilde{\circ}_C\,\hat{t}_C \vdash X}{\hat{\top} \vdash X} \qquad \frac{X \vdash \tilde{\circ}_C\,\check{t}_C}{X \vdash \check{\bot}}\ \tilde{\circ}_C\,\check{t}_C$$

$$\hat{\blacklozenge}_I\tilde{\circ}_I\ \frac{\hat{\blacklozenge}_I\tilde{\circ}_I\Gamma \vdash \Gamma'}{\Gamma \vdash \Gamma'} \qquad \frac{\Gamma' \vdash \check{\blacksquare}_I\tilde{\circ}_I\Gamma}{\Gamma' \vdash \Gamma}\ \check{\blacksquare}_I\tilde{\circ}_I \qquad \hat{\blacklozenge}_C\tilde{\circ}_C\ \frac{\hat{\blacklozenge}_C\tilde{\circ}_C\Delta \vdash \Delta'}{\Delta \vdash \Delta'} \qquad \frac{\Delta' \vdash \check{\blacksquare}_C\tilde{\circ}_C\Delta}{\Delta' \vdash \Delta}$$

$$\tilde{\circ}_I\,\hat{\blacklozenge}_I\ \frac{\tilde{\circ}_I\,\hat{\blacklozenge}_I X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash \tilde{\circ}_I\,\check{\blacksquare}_I X}{Y \vdash X}\ \tilde{\circ}_I\,\check{\blacksquare}_I \qquad \tilde{\circ}_C\,\hat{\blacklozenge}_C\ \frac{\tilde{\circ}_C\,\hat{\blacklozenge}_C X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash \tilde{\circ}_C\,\check{\blacksquare}_C X}{Y \vdash X}\ \tilde{\circ}_C$$

- Multi-type structural rules for lax-kernel operators:

$$\tilde{\bullet}_I\,\hat{1}_I\ \frac{\tilde{\bullet}_I\,\hat{\top} \vdash \Pi}{\hat{1}_I \vdash \Pi} \qquad \frac{\Pi \vdash \tilde{\bullet}_I\,\check{\bot}}{\Pi \vdash \check{0}_I}\ \tilde{\bullet}_I\,\check{0}_I \qquad \tilde{\bullet}\,\hat{1}_C\ \frac{\tilde{\bullet}_C\,\hat{\top} \vdash \Sigma}{\hat{1}_C \vdash \Sigma} \qquad \frac{\Sigma \vdash \tilde{\bullet}_C\,\check{\bot}}{\Sigma \vdash \check{0}_C}\ \tilde{\bullet}_C$$

$$\hat{\Diamond}_I\,\tilde{\bullet}_I\ \frac{\Pi \vdash \Pi'}{\hat{\Diamond}_I\,\tilde{\bullet}_I\Pi \vdash \Pi'} \qquad \frac{\Pi' \vdash \Pi}{\Pi' \vdash \check{\Box}_I\,\tilde{\bullet}_I\Pi}\ \check{\Box}_I\,\tilde{\bullet}_I \qquad \hat{\Diamond}_C\,\tilde{\bullet}_C\ \frac{\Sigma \vdash \Sigma'}{\hat{\Diamond}_C\,\tilde{\bullet}_C\Sigma \vdash \Sigma'} \qquad \frac{\Sigma' \vdash \Sigma}{\Sigma' \vdash \check{\Box}_C\,\tilde{\bullet}_C\Sigma}$$

$$\tilde{\bullet}_I\,\hat{\Diamond}_I\ \frac{\tilde{\bullet}_I\,\hat{\Diamond}_I\Pi \vdash \Pi'}{\Pi \vdash \Pi'} \qquad \frac{\Pi' \vdash \tilde{\bullet}_I\,\check{\Box}_I\Pi}{\Pi' \vdash \Pi}\ \tilde{\bullet}_I\,\check{\Box}_I \qquad \tilde{\bullet}_C\,\hat{\Diamond}_C\ \frac{\tilde{\bullet}_C\,\hat{\Diamond}_C\Sigma \vdash \Sigma'}{\Sigma \vdash \Sigma'} \qquad \frac{\Sigma' \vdash \tilde{\bullet}_C\,\check{\Box}_C\Sigma}{\Sigma' \vdash \Sigma}$$

Multi-type calculi

- Multi-type structural rules for the correspondence between kernels:

$$\tilde{\circ}\,\hat{\blacklozenge}\ \frac{\tilde{\circ}_I\,\hat{\blacklozenge}_I X \vdash Y}{\tilde{\circ}_C\,\hat{\blacklozenge}_C X \vdash Y} \qquad \frac{Y \vdash \check{\Box}_I\,\tilde{\bullet}_I X}{Y \vdash \check{\Box}_C\,\tilde{\bullet}_C X}\ \check{\Box}\,\tilde{\bullet}$$

- Logical rules for multi-type connectives related to strict kernels:

$$\blacklozenge_I\ \frac{\hat{\blacklozenge}_I A \vdash \Gamma}{\blacklozenge_I A \vdash \Gamma} \qquad \frac{X \vdash A}{\hat{\blacklozenge}_I X \vdash \blacklozenge_I A}\ \blacklozenge_I \qquad \blacksquare_C\ \frac{A \vdash X}{\blacksquare_C A \vdash \check{\blacksquare}_C X} \qquad \frac{\Delta \vdash \check{\blacksquare}_C A}{\Delta \vdash \blacksquare_C A}\ \blacksquare_C$$

$$\circ_I\ \frac{\tilde{\circ}_I \alpha \vdash X}{\circ_I \alpha \vdash X} \qquad \frac{X \vdash \tilde{\circ}_I \alpha}{X \vdash \circ_I \alpha}\ \circ_I \qquad \circ_C\ \frac{\tilde{\circ}_C \delta \vdash X}{\circ_C \delta \vdash X} \qquad \frac{X \vdash \tilde{\circ}_C \delta}{X \vdash \circ_C \delta}\ \circ_C$$

Multi-type calculi

- Logical rules for multi-type connectives related to lax kernels:

$$\diamond_I \frac{\hat{\diamond}_I \pi \vdash X}{\diamond_I \pi \vdash X} \qquad \frac{\Pi \vdash \pi}{\hat{\diamond}_I \Pi \vdash \diamond_I \pi} \diamond_I \qquad \Box_C \frac{\sigma \vdash \Sigma}{\Box_C \sigma \vdash \check{\Box}_C \Sigma} \qquad \frac{X \vdash \check{\Box}_C \sigma}{X \vdash \Box_C \sigma} \Box_C$$

$$\circ_I \frac{\tilde{\bullet}_I A \vdash \Pi}{\bullet_I A \vdash \Pi} \qquad \frac{\Pi \vdash \tilde{\bullet}_I A}{\Pi \vdash \bullet_I A} \bullet_I \qquad \bullet_C \frac{\tilde{\bullet}_C A \vdash \Sigma}{\bullet_C A \vdash \Sigma} \qquad \frac{\Sigma \vdash \tilde{\bullet}_C A}{\Sigma \vdash \bullet_C A} \bullet_C$$

Multi-type calculi

The proper display calculi for the subvarieties of *aKa* are obtained by adding the following rules:

Logic	Calculus	Rules
$H.aKa5'$	$D.aKa5'$	$\tilde{\circ}_I \hat{\blacklozenge}_I \hat{\Diamond}_I \dfrac{\hat{\Diamond}_I \Pi \vdash X}{\tilde{\circ}_I \hat{\blacklozenge}_I \hat{\Diamond}_I \Pi \vdash X}$ $\quad \dfrac{X \vdash \check{\Box}_C \Sigma}{X \vdash \tilde{\circ}_C \check{\blacksquare}_C \check{\Box}_C \Sigma} \tilde{\circ}_C \check{\blacksquare}_C \check{\Box}_C$ $\hat{\Diamond}_I \tilde{\bullet}_I \tilde{\circ}_I \dfrac{\hat{\Diamond}_I \tilde{\bullet}_I \tilde{\circ}_I \Gamma \vdash X}{\tilde{\circ}_I \Gamma \vdash X}$ $\quad \dfrac{X \vdash \check{\Box}_C \tilde{\bullet}_C \tilde{\circ}_C \Delta}{X \vdash \tilde{\circ}_C \Delta} \check{\Box}_C \tilde{\bullet}_C \tilde{\circ}_C$
$K\text{-}IA3_\ell$	$D.K\text{-}IA3_\ell$	$\dfrac{X \vdash \check{\Box}_I \tilde{\bullet}_I Y \qquad \hat{\Diamond}_C \tilde{\bullet}_C X \vdash Y}{X \vdash Y} \; k\text{-}ia3_\ell$

Properties

Theorem (Soundness)

The rules in D.K is sound w.r.t. the class of HD.

Theorem (Completeness)

D.K is complete with respect to the class of semi-De Morgan algebras.

Theorem (Conservativity)

D.K is a conservative extension of H.K.

Theorem (Cut elimination)

If $X \vdash Y$ is derivable in D.K, then it is derivable without (Cut).

Theorem (Subformula property)

Any cut-free proof of the sequent $X \vdash Y$ in D.K contains only structures over subformulas of formulas in X and Y.

Thank You!

References

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