A Paraconsistent Sub-Logic of Intuitionistic Propositional Logic

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Scheme

Scheme

Paraconsistency

Definition

A *theory* (i.e., a set of sentences closed under some deductive relation) is said to be (negation) *consistent* if for no sentence α is both α and $\neg\alpha$ provable, else *inconsistent*.
A theory is said to be *non-trivial* if not every formula is provable, else *trivial*.

Definition

A logic is said to be *paraconsistent* or *inconsistency tolerant* if it admits inconsistent but non-trivial theories.

In other words, a paraconsistent logic is a logic where it is not always possible to derive everything from a contradiction.

Paraconsistency

Classical logic, and also many non-classical logics, such as intuitionistic logic, fail in this because of the so-called principle of *'explosion'* by which, for any sentence α,

$$\{\alpha, \neg\alpha\} \vdash \beta \qquad (\text{ECQ}: \textit{ Ex contradictione quodlibet})$$

Thus a necessary condition for a logic to be paraconsistent is that its consequence relation be not explosive, thus invalidating ECQ.

Scheme

Stage Setting

Definition

Given a similarity type ν, the absolutely free algebra $\mathbf{Fm}$ of type ν over a countably infinite set X of generators is called the *formula algebra* of type ν; its underlying set will be denoted by Fm.

The elements of Fm are called *ν-terms* or *ν-formulas* and referred to by the symbols $t, s, \ldots$ or $\alpha, \beta, \varphi, \ldots$.

Members of X are called (propositional) *variables* and denoted by the symbols $x, y, \ldots$ or $p, q, \ldots$.

Definition

A *logic* of type ν is a pair $\mathrm{L} = \langle \mathbf{Fm}, \vdash_{\mathrm{L}} \rangle$, where $\mathbf{Fm}$ is the formula algebra of type ν, and $\vdash_{\mathrm{L}}$ is the substitution-invariant consequence relation over $\mathbf{Fm}$.

Paraconsistent Intuitionistic Logic (PIL)

Paraconsistent intuitionistic logic PIL $= \langle \mathbf{Fm}, \models^{\#} \rangle$ can be semantically specified as follows.

- $\mathbf{Fm}$ is the formula algebra of type $(2, 2, 2, 1, 0, 0)$, namely, of the type containing the connectives $\wedge, \vee, \longrightarrow, \neg, 0, 1$.
- For any Heyting algebra $\mathbf{A}$, we define its extension $\mathbf{A}^{\#} = \mathbf{A} \cup \{\omega\}$, where $\omega \notin \mathbf{A}$ for all Heyting algebras $\mathbf{A}$. Let $\mathcal{S} = \{\mathbf{A}^{\#} \mid \mathbf{A}$ is a Heyting algebra$\}$.
- The additional element ω satisfies the so-called *contamination priniciple*, that is, $\neg\omega = \omega$ and if $\mathbf{A}$ is any Heyting algebra, then for any $a \in \mathbf{A}$, $a \circ \omega = \omega$, where $\circ$ denotes any of the binary connectives $\wedge, \vee, \longrightarrow$.
- Finally, for any $\Sigma \cup \{\alpha\} \subseteq Fm$, we define $\Sigma \models^{\#} \alpha \iff$ for every $\mathbf{A}^{\#} \in \mathcal{S}$ and for every valuation $v^{\#} : Fm \to \mathbf{A}^{\#}$, $v^{\#}[\Sigma] \subseteq \{1, \omega\}$ implies $v^{\#}(\alpha) \in \{1, \omega\}$.

Relating $\models^{\#}$ and $\models_{\mathrm{IL}}$

Theorem

For all $\Sigma \cup \{\alpha\} \subseteq Fm$, we have that $\Sigma \models^{\#} \alpha$ if and only if there is a $\Delta \subseteq \Sigma$ such that $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\alpha)$ and $\Delta \models_{\mathrm{IL}} \alpha$. Moreover, since IL (intuitionistic propositional logic) is finitary, we can find such a $\Delta \subseteq \Sigma$ that is finite.

Proof Outline

- Suppose $\Sigma \models^{\#} \alpha$. Let $\Delta = \{\varphi \in \Sigma \mid \mathrm{var}(\varphi) \subseteq \mathrm{var}(\alpha)\}$.
- Suppose $\Delta \not\models_{\mathrm{IL}} \alpha$. So there exists a Heyting algebra $\mathbf{A}$ and a valuation $v : Fm \to \mathbf{A}$ such that $v[\Delta] = \{1\}$ but $v(\alpha) \neq 1$.
- We construct the valuation $v^{\#} : Fm \to \mathbf{A}^{\#}$ as follows.

$$v^{\#}(p) = \begin{cases} v(p) & \text{if } p \in \mathrm{var}(\alpha) \\ \omega & \text{if } p \notin \mathrm{var}(\alpha) \end{cases}$$

- For any $\varphi \in \Sigma$, either $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\alpha)$ or $\mathrm{var}(\varphi) \cap (\mathrm{var}(\Sigma) \setminus \mathrm{var}(\alpha)) \neq \emptyset$.
- In the first case, $v^{\#}(\varphi) = 1$, and in the second case, $v^{\#}(\varphi) = \omega$.
- Thus $v^{\#}[\Sigma] \subseteq \{1, \omega\}$. Then $v^{\#}(\alpha) \in \{1, \omega\}$.
- Now, $v^{\#}(\alpha) = v(\alpha) \in \mathbf{A}$ and $v(\alpha) \neq 1$. So $v^{\#}(\alpha) \notin \{1, \omega\}$. This is a contradiction, which proves that $\Delta \models_{\mathrm{IL}} \alpha$.

Relating $\models^{\#}$ and $\models_{\text{IL}}$

Proof Outline

- Conversely, suppose $\Delta \models_{\text{IL}} \alpha$ for some $\Delta \subseteq \Sigma$ with $\text{var}(\Delta) \subseteq \text{var}(\alpha)$.
- Suppose $v^{\#} : Fm \to \mathbf{A}^{\#}$ for some $\mathbf{A}^{\#} \in \mathcal{S}$ be a valuation such that $v^{\#}[\Sigma] \subseteq \{1, \omega\}$.
- The possible cases are $v^{\#}(p) = \omega$ for some $p \in \text{var}(\alpha)$ or $v^{\#}(p) \neq \omega$ for all $p \in \text{var}(\alpha)$.
- In the first case, $v^{\#}(\alpha) = \omega$.
- In the second case, let $\mathbf{A}$ be the Heyting algebra corresponding to $\mathbf{A}^{\#} \in \mathcal{S}$ and $a_0 \in \mathbf{A}$ (fixed).
  We construct a valuation $v : Fm \to \mathbf{A}$ as follows.

$$v(p) = \begin{cases} v^{\#}(p) & \text{if } p \in \text{var}(\alpha) \\ a_0 & \text{otherwise} \end{cases}$$

- Then, since $\text{var}(\Delta) \subseteq \text{var}(\alpha)$, $v[\Delta] = v^{\#}[\Delta] \subseteq v^{\#}[\Sigma] \subseteq \{1, \omega\}$. So $v[\Delta] \subseteq \{1, \omega\} \cap \mathbf{A} = \{1\}$. Thus $v^{\#}(\alpha) = v(\alpha) = 1 \in \{1, \omega\}$.
  This proves that $\Sigma \models^{\#} \alpha$.

$\square$

Scheme

An axiomatization of PIL

Definition (HPIL)

HPIL is the logic $\langle \mathbf{Fm}, \vdash^{\#} \rangle$, where $\vdash^{\#}$ is the consequence relation of the deductive system with the following axioms and inference rule.

1. $\alpha \longrightarrow (\beta \longrightarrow \alpha)$;
2. $(\alpha \longrightarrow (\beta \longrightarrow \gamma)) \longrightarrow ((\alpha \longrightarrow \beta) \longrightarrow (\alpha \longrightarrow \gamma))$;
3. $\alpha \longrightarrow (\beta \longrightarrow \alpha \wedge \beta)$;
4. $\alpha \wedge \beta \longrightarrow \alpha$;
5. $\alpha \wedge \beta \longrightarrow \beta$;
6. $\alpha \longrightarrow \alpha \vee \beta$;
7. $\beta \longrightarrow \alpha \vee \beta$;
8. $(\alpha \longrightarrow \gamma) \longrightarrow ((\beta \longrightarrow \gamma) \longrightarrow (\alpha \vee \beta \longrightarrow \gamma))$;
9. $(\alpha \longrightarrow \beta) \longrightarrow ((\alpha \longrightarrow \neg \beta) \longrightarrow \neg \alpha)$;
10. $0 \longrightarrow \alpha$;

$$[\text{RMP}] \quad \frac{\alpha, \quad \alpha \longrightarrow \beta}{\beta} \quad \text{provided that } \text{var}(\alpha) \subseteq \text{var}(\beta).$$

Connections with IL

Remark

- Axioms (1)–(10) along with unrestricted modus ponens gives a Hilbert system for intuitionistic propositional logic (IL).
- It is easy to check that PIL fails to satisfy modus ponens (MP). However, it satisfies RMP as we will see later.
- Because of the restricted nature of the inference relation, HPIL is weaker than IL.

However, the following result shows that both logics have the same theorems.

Theorem

For any $\varphi \in Fm$, $\vdash^{\#} \varphi$ if and only if $\vdash_{\mathrm{IL}} \varphi$.

Connections with IL

Proof Outline

($\Longrightarrow$)
Immediate, since the axioms (1)–(10) are axioms of IL and RMP is also an instance of MP.
($\Longleftarrow$)

- Suppose $\vdash_{\text{IL}} \varphi$ and $D = \langle \varphi_1, \ldots, \varphi_n \rangle$ is a proof of φ in IL.
- We use induction on the length n of D.
- In the inductive step, suppose $\varphi = \varphi_l$ is obtained by MP from φ_i and $\varphi_j = \varphi_i \longrightarrow \varphi$, where $i, j < l$.
- By the induction hypothesis, there are proofs of φ_i, φ_j in HPIL. Let $\langle \psi_1, \ldots, \psi_m \rangle$ be the result of gluing these HPIL proofs so that $\psi_k = \varphi_i (k < m)$ and $\psi_m = \varphi_j = \varphi_i \longrightarrow \varphi$.

Connections with IL

Proof Outline

- We use a substitution of variables: $\sigma(p) = \begin{cases} p & \text{if } p \in \text{var}(\varphi) \\ a & \text{otherwise} \end{cases}$, where a is some fixed variable in φ or 0 if $\text{var}(\varphi) = \emptyset$.
- Note that $\langle \sigma(\psi_1), \ldots, \sigma(\psi_m) \rangle$ is still a proof in HPIL.
- Now, $\sigma(\psi_m) = \sigma(\psi_k \longrightarrow \varphi) = \sigma(\psi_k) \longrightarrow \sigma(\varphi) = \sigma(\psi_k) \longrightarrow \varphi$. Moreover, $\text{var}(\sigma(\psi_k)) \subseteq \text{var}(\varphi)$, by definition of σ.
- Thus φ follows from $\sigma(\psi_k)$ and $\sigma(\psi_m)$ by an application of RMP, and $\langle \sigma(\psi_1), \ldots, \sigma(\psi_k), \ldots, \sigma(\psi_m), \varphi \rangle$ is a proof of φ in HPIL.

$\square$

Restricted Deduction Theorem

Theorem (Restricted Deduction Theorem)

For any $\Sigma \cup \{\alpha, \beta\} \subseteq Fm$, we have the following.

(i) $\Sigma \cup \{\alpha\} \vdash^{\#} \beta$ implies $\Sigma \vdash^{\#} \alpha \longrightarrow \beta$. In particular, if $\alpha \vdash^{\#} \beta$ then $\vdash^{\#} \alpha \longrightarrow \beta$.

(ii) If $\Sigma \vdash^{\#} \alpha \longrightarrow \beta$ and $\mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$, then $\Sigma \cup \{\alpha\} \vdash^{\#} \beta$. In particular, if $\vdash^{\#} \alpha \longrightarrow \beta$ and $\mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$, then $\alpha \vdash^{\#} \beta$.

Scheme

Soundness

Lemma

For any $\varphi \in Fm$, if $\vdash^{\#} \varphi$, then $\models^{\#} \varphi$.

Proof Outline

- Let $\mathbf{A}^{\#} \in \mathcal{S}$ and $v^{\#} : Fm \to \mathbf{A}^{\#}$ be any valuation.
- If for some variable $p \in \text{var}(\varphi)$, $v^{\#}(p) = \omega$, then $v^{\#}(\varphi) = \omega$, and so $\models^{\#} \varphi$.
- If on the other hand, for all $p \in \text{var}(\varphi)$, $v^{\#}(p) \neq \omega$, then $v^{\#}(p) \in \mathbf{A}$, the Heyting algebra corresponding to $\mathbf{A}^{\#}$, for all $p \in \text{var}(\varphi)$. This implies that $v^{\#}(\varphi) \in \mathbf{A}$. We construct the valuation $v : Fm \to \mathbf{A}$ as follows.

$$v(p) = \begin{cases} v^{\#}(p) & \text{if } p \in \text{var}(\varphi) \\ a_0 & \text{otherwise,} \end{cases} \text{, where } a_0 \in \mathbf{A} \text{ (fixed)}$$

- Then $v(\varphi) = v^{\#}(\varphi)$. Now, since $\vdash^{\#} \varphi$, $\vdash_{\text{IL}} \varphi$ which implies that $v^{\#}(\varphi) = v(\varphi) = 1$. Hence $\models^{\#} \varphi$.

$\square$

Soundness

Lemma

Suppose $\varphi, \psi \in Fm$ with $\text{var}(\varphi) \subseteq \text{var}(\psi)$. Then for any $\mathbf{A}^{\#} \in \mathcal{S}$, and any valuation $v^{\#} : Fm \to \mathbf{A}^{\#}$, if $v^{\#}[\{\varphi, \varphi \longrightarrow \psi\}] \subseteq \{1, \omega\}$, then $v^{\#}(\psi) \in \{1, \omega\}$. That is, RMP preserves validity.

Proof Outline

- Suppose not. Then $v^{\#}[\{\varphi, \varphi \longrightarrow \psi\}] \subseteq \{1, \omega\}$, but $v^{\#}(\psi) \notin \{1, \omega\}$.
- So $v^{\#}(p) \neq \omega$ for all $p \in \text{var}(\psi)$. Then $v^{\#}(\psi) \in \mathbf{A}$, the Heyting algebra corresponding to $\mathbf{A}^{\#}$.
- Let $v^{\#}(\psi) = a \in \mathbf{A}$, where $a < 1$.
- Since $\text{var}(\varphi) \subseteq \text{var}(\psi)$, $v^{\#}(p) \neq \omega$ for all $p \in \text{var}(\varphi)$, and hence $v^{\#}(\varphi), v^{\#}(\varphi \longrightarrow \psi) = 1$.
- Then $v^{\#}(\varphi \longrightarrow \psi) = 1 \longrightarrow a$.
- For any $a \in \mathbf{A}$, $1 \longrightarrow a = a$, $v^{\#}(\varphi \longrightarrow \psi) = a < 1$. This is a contradiction.

$\square$

Soundness

Theorem (Soundness)

For all $\Sigma \cup \{\alpha\} \subseteq Fm$, if $\Sigma \vdash^{\#} \alpha$, then $\Sigma \models^{\#} \alpha$.

Proof

Straightforward from the two lemmas.

Completeness

Theorem

For any $\varphi \in Fm$, if $\models^{\#} \varphi$ then $\vdash^{\#} \varphi$.

Proof.

- Suppose $\models^{\#} \varphi$.
- Then $\models_{\text{IL}} \varphi$.
- We know that IL is complete with respect to valuations in Heyting algebras. So $\vdash_{\text{IL}} \varphi$.
- Hence $\vdash^{\#} \varphi$.

□

Completeness

Theorem (Completeness)

For all $\Sigma \cup \{\alpha\} \subseteq Fm$, if $\Sigma \models^{\#} \alpha$, then $\Sigma \vdash^{\#} \alpha$.

Proof.

- Suppose $\Sigma \models^{\#} \alpha$ and $\Sigma \neq \emptyset$.
- Then there exists a finite $\Delta \subseteq \Sigma$ with $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\alpha)$ such that $\Delta \models_{\mathrm{IL}} \alpha$.
- Now, by the completeness of IL with respect to valuations in Heyting algebras, $\Delta \vdash_{\mathrm{IL}} \alpha$.
- Let $\Delta = \{\varphi_1, \ldots, \varphi_n\}$. Then by applying the deduction theorem in IL n times, we have $\vdash_{\mathrm{IL}} (\varphi_1 \longrightarrow (\ldots \longrightarrow (\varphi_n \longrightarrow \alpha)\ldots))$.
- Then $\vdash^{\#} (\varphi_1 \longrightarrow (\ldots \longrightarrow (\varphi_n \longrightarrow \alpha)\ldots))$.
- Now, since $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\alpha)$, by applying the converse of the Deduction theorem n times, we have $\Delta \vdash^{\#} \alpha$.
- Finally, since $\Delta \subseteq \Sigma$, we have $\Sigma \vdash^{\#} \alpha$.

□

Scheme

Concluding Remarks

- ECQ fails in HPIL. To see this, consider $\alpha = p$ and $\beta = q$ where p, q are variables, and a valuation $v^{\#} : Fm \to A^{\#}$, where $A^{\#} \in \mathcal{S}$, such that $v^{\#}(p) = \omega$ and $v^{\#}(q) = 0$. Then $\{\alpha, \neg\alpha\} \not\models^{\#} \beta$. Then, by the Soundness Theorem, $\{\alpha, \neg\alpha\} \nvdash^{\#} \beta$.
  Thus HPIL is paraconsistent.
- Interestingly, however, $0 \longrightarrow \alpha$ is an axiom of HPIL.
- For any $\alpha \in Fm$, $\neg(\alpha \wedge \neg\alpha)$ is a theorem of IL and hence of HPIL. Thus the law of non-contradiction holds in HPIL.
- However, it is noteworthy that the sentence $\alpha \wedge \neg\alpha$ is not always false either. To see this, take $\alpha = p$, a variable and a valuation $v^{\#}$ such that $v^{\#}(p) = \omega$.

Thank You