A Paraconsistent Sub-Logic of Intuitionistic Propositional Logic

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Paraconsistency

Definition

A *theory* (i.e., a set of sentences closed under some deductive relation) is said to be (negation) *consistent* if for no sentence $\alpha$ is both $\alpha$ and $\neg \alpha$ provable, else *inconsistent*. A theory is said to be *non-trivial* if not every formula is provable, else *trivial*.

Definition

A logic is said to be *paraconsistent* or *inconsistency tolerant* if it admits inconsistent but non-trivial theories.

In other words, a paraconsistent logic is a logic where it is not always possible to derive everything from a contradiction.
Paraconsistency

Classical logic, and also many non-classical logics, such as intuitionistic logic, fail in this because of the so-called principle of ‘explosion’ by which, for any sentence \( \alpha \),

\[ \{\alpha, \neg \alpha\} \vdash \beta \quad \text{(ECQ : Ex contradictione quodlibet)} \]

Thus a necessary condition for a logic to be paraconsistent is that its consequence relation be not explosive, thus invalidating ECQ.
Scheme

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Stage Setting

Definition

Given a similarity type $\nu$, the absolutely free algebra $Fm$ of type $\nu$ over a countably infinite set $X$ of generators is called the *formula algebra* of type $\nu$; its underlying set will be denoted by $Fm$.

The elements of $Fm$ are called *$\nu$-terms* or *$\nu$-formulas* and referred to by the symbols $t, s, \ldots$ or $\alpha, \beta, \varphi, \ldots$.

Members of $X$ are called (propositional) *variables* and denoted by the symbols $x, y, \ldots$ or $p, q, \ldots$.

Definition

A *logic* of type $\nu$ is a pair $L = \langle Fm, \vdash_L \rangle$, where $Fm$ is the formula algebra of type $\nu$, and $\vdash_L$ is the substitution-invariant consequence relation over $Fm$. 
Paraconsistent Intuitionistic Logic (PIL)

Paraconsistent intuitionistic logic PIL = \langle Fm, \models \# \rangle can be semantically specified as follows.

- **Fm** is the formula algebra of type \((2, 2, 2, 1, 0, 0)\), namely, of the type containing the connectives \(\land, \lor, \rightarrow, \neg, 0, 1\).
- For any Heyting algebra \(A\), we define its extension \(A^\# = A \cup \{\omega\}\), where \(\omega \notin A\) for all Heyting algebras \(A\). Let \(S = \{A^\# | A\text{ is a Heyting algebra}\}\).
- The additional element \(\omega\) satisfies the so-called contamination principle, that is, \(\neg \omega = \omega\) and if \(A\) is any Heyting algebra, then for any \(a \in A\), \(a \circ \omega = \omega\), where \(\circ\) denotes any of the binary connectives \(\land, \lor, \rightarrow\).
- Finally, for any \(\Sigma \cup \{\alpha\} \subseteq Fm\), we define \(\Sigma \models \# \alpha \iff\) for every \(A^\# \in S\) and for every valuation \(v^\#: Fm \rightarrow A^\#\), \(v^\#[\Sigma] \subseteq \{1, \omega\}\) implies \(v^\#(\alpha) \in \{1, \omega\}\).
Relating $\models \#$ and $\models_{\text{IL}}$

**Theorem**

For all $\Sigma \cup \{\alpha\} \subseteq \text{Fm}$, we have that $\Sigma \models \# \alpha$ if and only if there is a $\Delta \subseteq \Sigma$ such that $\text{var}(\Delta) \subseteq \text{var}(\alpha)$ and $\Delta \models_{\text{IL}} \alpha$. Moreover, since IL (intuitionistic propositional logic) is finitary, we can find such a $\Delta \subseteq \Sigma$ that is finite.

**Proof Outline**

- Suppose $\Sigma \models \# \alpha$. Let $\Delta = \{\varphi \in \Sigma \mid \text{var}(\varphi) \subseteq \text{var}(\alpha)\}$.
- Suppose $\Delta \not\models_{\text{IL}} \alpha$. So there exists a Heyting algebra $\mathbf{A}$ and a valuation $v : \text{Fm} \to \mathbf{A}$ such that $v[\Delta] = \{1\}$ but $v(\alpha) \neq 1$.
- We construct the valuation $v^\# : \text{Fm} \to \mathbf{A}^\#$ as follows.
  
  $$v^\#(p) = \begin{cases} 
  v(p) & \text{if } p \in \text{var}(\alpha) \\
  \omega & \text{if } p \notin \text{var}(\alpha)
  \end{cases}$$

- For any $\varphi \in \Sigma$, either $\text{var}(\varphi) \subseteq \text{var}(\alpha)$ or $\text{var}(\varphi) \cap (\text{var}(\Sigma) \setminus \text{var}(\alpha)) \neq \emptyset$.
- In the first case, $v^\#(\varphi) = 1$, and in the second case, $v^\#(\varphi) = \omega$.
- Thus $v^\#[\Sigma] \subseteq \{1, \omega\}$. Then $v^\#(\alpha) \in \{1, \omega\}$.
- Now, $v^\#(\alpha) = v(\alpha) \in \mathbf{A}$ and $v(\alpha) \neq 1$. So $v^\#(\alpha) \notin \{1, \omega\}$. This is a contradiction, which proves that $\Delta \models_{\text{IL}} \alpha$. 

Relating $\models \# \text{ and } \models_{\text{IL}}$

**Proof Outline**

- Conversely, suppose $\Delta \models_{\text{IL}} \alpha$ for some $\Delta \subseteq \Sigma$ with $\text{var}(\Delta) \subseteq \text{var}(\alpha)$.
- Suppose $v^\# : Fm \to A^\#$ for some $A^\# \in S$ be a valuation such that $v^\#[\Sigma] \subseteq \{1, \omega\}$.
- The possible cases are $v^\#(p) = \omega$ for some $p \in \text{var}(\alpha)$ or $v^\#(p) \neq \omega$ for all $p \in \text{var}(\alpha)$.
- In the first case, $v^\#(\alpha) = \omega$.
- In the second case, let $A$ be the Heyting algebra corresponding to $A^\# \in S$ and $a_0 \in A$ (fixed).
  We construct a valuation $v : Fm \to A$ as follows.

\[
v(p) = \begin{cases} 
  v^\#(p) & \text{if } p \in \text{var}(\alpha) \\
  a_0 & \text{otherwise}
\end{cases}
\]

- Then, since $\text{var}(\Delta) \subseteq \text{var}(\alpha)$, $v[\Delta] = v^\#[\Delta] \subseteq v^\#[\Sigma] \subseteq \{1, \omega\}$. So $v[\Delta] \subseteq \{1, \omega\} \cap A = \{1\}$. Thus $v^\#(\alpha) = v(\alpha) = 1 \in \{1, \omega\}$. This proves that $\Sigma \models \# \alpha$. 
Scheme

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An axiomatization of PIL

Definition (HPIL)

HPIL is the logic $\langle \mathbf{Fm}, \vdash \# \rangle$, where $\vdash \#$ is the consequence relation of the deductive system with the following axioms and inference rule.

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$;
2. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$;
3. $\alpha \rightarrow (\beta \rightarrow \alpha \land \beta)$;
4. $\alpha \land \beta \rightarrow \alpha$;
5. $\alpha \land \beta \rightarrow \beta$;
6. $\alpha \rightarrow \alpha \lor \beta$;
7. $\beta \rightarrow \alpha \lor \beta$;
8. $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \lor \beta \rightarrow \gamma))$;
9. $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg \beta) \rightarrow \neg \alpha)$;
10. $0 \rightarrow \alpha$;

[RMP] \[ \frac{\alpha, \alpha \rightarrow \beta}{\beta} \] provided that $\text{var}(\alpha) \subseteq \text{var}(\beta)$. 
Connections with IL

Remark

- Axioms (1)–(10) along with unrestricted modus ponens gives a Hilbert system for intuitionistic propositional logic (IL).
- It is easy to check that PIL fails to satisfy modus ponens (MP). However, it satisfies RMP as we will see later.
- Because of the restricted nature of the inference relation, HPIL is weaker than IL.

However, the following result shows that both logics have the same theorems.

Theorem

For any $\varphi \in Fm$, $\vdash \# \varphi$ if and only if $\vdash_{IL} \varphi$. 
Connections with IL

Proof Outline

$(\implies)$ Immediate, since the axioms (1)–(10) are axioms of IL and RMP is also an instance of MP.

$(\impliedby)$

- Suppose $\vdash_{\text{IL}} \varphi$ and $D = \langle \varphi_1, \ldots, \varphi_n \rangle$ is a proof of $\varphi$ in IL.
- We use induction on the length $n$ of $D$.
- In the inductive step, suppose $\varphi = \varphi_l$ is obtained by MP from $\varphi_i$ and $\varphi_j = \varphi_i \rightarrow \varphi$, where $i, j < l$.
- By the induction hypothesis, there are proofs of $\varphi_i, \varphi_j$ in HPIL. Let $\langle \psi_1, \ldots, \psi_m \rangle$ be the result of gluing these HPIL proofs so that $\psi_k = \varphi_i (k < m)$ and $\psi_m = \varphi_j = \varphi_i \rightarrow \varphi$. 
Connections with IL

Proof Outline

- We use a substitution of variables: \( \sigma(p) = \begin{cases} p & \text{if } p \in \text{var}(\varphi) \\ a & \text{otherwise} \end{cases} \), where \( a \) is some fixed variable in \( \varphi \) or 0 if \( \text{var}(\varphi) = \emptyset \).
- Note that \( \langle \sigma(\psi_1), \ldots, \sigma(\psi_m) \rangle \) is still a proof in HPIL.
- Now, \( \sigma(\psi_m) = \sigma(\psi_k \rightarrow \varphi) = \sigma(\psi_k) \rightarrow \sigma(\varphi) = \sigma(\psi_k) \rightarrow \varphi \).
  Moreover, \( \text{var}(\sigma(\psi_k)) \subseteq \text{var}(\varphi) \), by definition of \( \sigma \).
- Thus \( \varphi \) follows from \( \sigma(\psi_k) \) and \( \sigma(\psi_m) \) by an application of RMP, and \( \langle \sigma(\psi_1), \ldots, \sigma(\psi_k), \ldots, \sigma(\psi_m), \varphi \rangle \) is a proof of \( \varphi \) in HPIL.
Theorem (Restricted Deduction Theorem)

For any $\Sigma \cup \{\alpha, \beta\} \subseteq \text{Fm}$, we have the following.

(i) $\Sigma \cup \{\alpha\} \vdash \# \beta$ implies $\Sigma \vdash \# \alpha \rightarrow \beta$. In particular, if $\alpha \vdash \# \beta$ then $\vdash \# \alpha \rightarrow \beta$.

(ii) If $\Sigma \vdash \# \alpha \rightarrow \beta$ and $\text{var}(\alpha) \subseteq \text{var}(\beta)$, then $\Sigma \cup \{\alpha\} \vdash \# \beta$. In particular, if $\vdash \# \alpha \rightarrow \beta$ and $\text{var}(\alpha) \subseteq \text{var}(\beta)$, then $\alpha \vdash \# \beta$. 
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Soundness

Lemma

For any $\varphi \in Fm$, if $\vdash \# \varphi$, then $\models \# \varphi$.

Proof Outline

- Let $A \# \in S$ and $v \# : Fm \rightarrow A \#$ be any valuation.
- If for some variable $p \in \text{var}(\varphi)$, $v \#(p) = \omega$, then $v \#(\varphi) = \omega$, and so $\models \# \varphi$.
- If on the other hand, for all $p \in \text{var}(\varphi)$, $v \#(p) \neq \omega$, then $v \#(p) \in A$, the Heyting algebra corresponding to $A \#$, for all $p \in \text{var}(\varphi)$. This implies that $v \#(\varphi) \in A$. We construct the valuation $v : Fm \rightarrow A$ as follows.

$$v(p) = \begin{cases} v \#(p) & \text{if } p \in \text{var}(\varphi) \\ a_0 & \text{otherwise,} \end{cases} \quad \text{where } a_0 \in A \ (\text{fixed})$$

- Then $v(\varphi) = v \#(\varphi)$. Now, since $\vdash \# \varphi$, $\vdash_{IL} \varphi$ which implies that $v \#(\varphi) = v(\varphi) = 1$. Hence $\models \# \varphi$. 
Soundness

Lemma

Suppose $\varphi, \psi \in Fm$ with $\text{var}(\varphi) \subseteq \text{var}(\psi)$. Then for any $A\# \in S$, and any valuation $v\# : Fm \rightarrow A\#$, if $v\#[\{\varphi, \varphi \rightarrow \psi\}] \subseteq \{1, \omega\}$, then $v\#(\psi) \in \{1, \omega\}$. That is, RMP preserves validity.

Proof Outline

- Suppose not. Then $v\#[\{\varphi, \varphi \rightarrow \psi\}] \subseteq \{1, \omega\}$, but $v\#(\psi) \notin \{1, \omega\}$.
- So $v\#(p) \neq \omega$ for all $p \in \text{var}(\psi)$. Then $v\#(\psi) \in A$, the Heyting algebra corresponding to $A\#$.
- Let $v\#(\psi) = a \in A$, where $a < 1$.
- Since $\text{var}(\varphi) \subseteq \text{var}(\psi)$, $v\#(p) \neq \omega$ for all $p \in \text{var}(\varphi)$, and hence $v\#(\varphi), v\#(\varphi \rightarrow \psi) = 1$.
- Then $v\#(\varphi \rightarrow \psi) = 1 \rightarrow a$.
- For any $a \in A$, $1 \rightarrow a = a$, $v\#(\varphi \rightarrow \psi) = a < 1$. This is a contradiction.
Soundness

Theorem (Soundness)
For all $\Sigma \cup \{\alpha\} \subseteq Fm$, if $\Sigma \vdash \# \alpha$, then $\Sigma \models \# \alpha$.

Proof
Straightforward from the two lemmas.
Completeness

Theorem

For any $\varphi \in Fm$, if $\models \# \varphi$ then $\vdash \# \varphi$.

Proof.

- Suppose $\models \# \varphi$.
- Then $\models_{IL} \varphi$.
- We know that IL is complete with respect to valuations in Heyting algebras. So $\vdash_{IL} \varphi$.
- Hence $\vdash \# \varphi$. 


Completeness

Theorem (Completeness)

For all $\Sigma \cup \{\alpha\} \subseteq Fm$, if $\Sigma \models \# \alpha$, then $\Sigma \vdash \# \alpha$.

Proof.

- Suppose $\Sigma \models \# \alpha$ and $\Sigma \neq \emptyset$.
- Then there exists a finite $\Delta \subseteq \Sigma$ with $\text{var}(\Delta) \subseteq \text{var}(\alpha)$ such that $\Delta \models_{IL} \alpha$.
- Now, by the completeness of $IL$ with respect to valuations in Heyting algebras, $\Delta \vdash_{IL} \alpha$.
- Let $\Delta = \{\varphi_1, \ldots, \varphi_n\}$. Then by applying the deduction theorem in $IL$ $n$ times, we have $\vdash_{IL} (\varphi_1 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \alpha)\ldots))$.
- Then $\vdash \# (\varphi_1 \rightarrow (\ldots \rightarrow (\varphi_n \rightarrow \alpha)\ldots))$.
- Now, since $\text{var}(\Delta) \subseteq \text{var}(\alpha)$, by applying the converse of the Deduction theorem $n$ times, we have $\Delta \vdash \# \alpha$.
- Finally, since $\Delta \subseteq \Sigma$, we have $\Sigma \vdash \# \alpha$. 
Scheme

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Concluding Remarks

- ECQ fails in HPIL. To see this, consider $\alpha = p$ and $\beta = q$ where $p, q$ are variables, and a valuation $v^\# : Fm \rightarrow A^\#$, where $A^\# \in S$, such that $v^\#(p) = \omega$ and $v^\#(q) = 0$. Then $\{\alpha, \neg\alpha\} \not\models^\# \beta$. Then, by the Soundness Theorem, $\{\alpha, \neg\alpha\} \nvdash^\# \beta$. Thus HPIL is paraconsistent.

- Interestingly, however, $0 \rightarrow \alpha$ is an axiom of HPIL.

- For any $\alpha \in Fm$, $\neg(\alpha \land \neg\alpha)$ is a theorem of IL and hence of HPIL. Thus the law of non-contradiction holds in HPIL.

- However, it is noteworthy that the sentence $\alpha \land \neg\alpha$ is not always false either. To see this, take $\alpha = p$, a variable and a valuation $v^\#$ such that $v^\#(p) = \omega$. 
Thank You