Sequent Calculi for Normal Update Logics

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Outline

Kripke Semantics of Modal Logic

$\Box\psi$: "It is necessary that ψ."

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid \Box\varphi.$$

- Let $M = (W, R, V)$ where $R \subseteq W \times W$.

$$M, w \models \Box\psi \iff \text{For all } v \; (wRv \text{ implies } M, v \models \psi).$$

- $\Box\psi$ is Gψ (ψ will be always the case) for Tense Logic

Kripke Semantics of Tense Logic

$\blacklozenge\,\psi$: "It was the case that ψ. " (Pψ)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \blacklozenge\,\varphi.$$

- Let $M = (W, R, V)$ where $R \subseteq W \times W$.

$$M, w \models \blacklozenge\,\psi \quad \Longleftrightarrow \quad \text{For some } v \ (vRw \text{ and } M, v \models \psi).$$

- $\models \blacklozenge\,\varphi \to \psi \Longleftrightarrow\ \models \varphi \to \Box\psi$.

Kripke Sem. of Conditional Logic (Chellas 1975)

$[\varphi]\psi$: "If φ then (normally) ψ." "The current belief base is updated by φ, ψ follows."

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid [\varphi]\varphi.$$

- Let $M = (W, (R_X)_{X \subseteq W}, V)$ where $R_X \subseteq W \times W$.

$$M, w \models [\varphi]\psi \iff \text{For all } v \ (wR_{[\![\varphi]\!]} v \text{ implies } M, v \models \psi),$$

$$\text{where } [\![\varphi]\!] := \{\, x \in W \mid M, x \models \varphi \,\}.$$

- $wR_X v$: v is one of the most "X-similar" states from w.

Kripke Sem. of Normal Update Logic

$\langle \varphi^{-} \rangle \psi$: "$\psi$ has been updated by φ." (cf. Herzig (1998))

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid [\varphi]\varphi \mid \langle \varphi^{-} \rangle \varphi.$$

- Let $M = (W, (R_X)_{X \subseteq W}, V)$ where $R_X \subseteq W \times W$.

$$M, w \models \langle \varphi^{-} \rangle \psi \iff \text{For some } v \ (v R_{[\![\varphi]\!]} w \text{ and } M, v \models \psi).$$

where $[\![\varphi]\!] := \{ x \in W \mid M, x \models \varphi \}$.

- $wR_X v$: v is one of the most "X-similar" states from w.

Motivation for Normal Update Logic

$\langle \varphi^{-} \rangle \psi$: "$\psi$ has been updated by φ." (cf. Herzig (1998))

Let p be an input, q a current belief base, r a resulting belief base.

- p = "I come to New Delhi"
- q = "I suffer from a heavy jet-lag in Japan,"
- r = "My jet-lag becomes lighter."

$$\langle p^{-} \rangle q \vdash r \quad \Longleftrightarrow \quad q \vdash [p]r$$

Herzig (1998), p.193

we neither consider updates to be more basic than conditionals nor the contrary, and shall rather take the equivalence to be basic.

H**CK** for Conditional Logic: Chellas 1975

(Taut) All instances of propositional tautologies

(MP) Modus Ponens

(K) $[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$

(Nec) From ψ we may infer $[\varphi]\psi$.

(EQCA) From $\varphi_1 \leftrightarrow \varphi_2$, we may infer $[\varphi_1]\psi \leftrightarrow [\varphi_2]\psi$.

H**UCK** for Normal Update Logic: Herzig 1998

To Hilbert system H**CK** for Conditional Logic, we add:

(Conv1) $\psi \to [\varphi]\langle\varphi^{-}\rangle\psi$.

(Conv2) $\langle\varphi^{-}\rangle[\varphi]\psi \to \psi$.

(Mon $\langle\cdot^{-}\rangle$) From $\psi_1 \to \psi_2$ we may infer $\langle\varphi^{-}\rangle\psi_1 \to \langle\varphi^{-}\rangle\psi_2$.

(EQUA) From $\varphi_1 \leftrightarrow \varphi_2$, we may infer $\langle\varphi_1^{-}\rangle\psi \leftrightarrow \langle\varphi_2^{-}\rangle\psi$.

$$\vdash_{\mathbf{HUCK}} \langle\varphi^{-}\rangle\psi \to \theta \iff \vdash_{\mathbf{HUCK}} \psi \to [\varphi]\theta$$

Semantic Completeness of HUCK** (Herzig 1998)**

φ is provable in H**UCK** iff φ is valid in all models.

($\because$) By Canonical Model Construction.

Background & Contribution of This Talk

Main Question of This Talk

Does H**UCK** enjoy the decidability or the finite model property (FMP)?

This talk

- provides sequent calculus G**UCK**;
- shows that it is equipollent with H**UCK**;
- establishes the subformula property and FMP of G**UCK** to obtain the decidability.

What are Sequents?

a sequent = a pair of finite sets of formulas

$$\varphi_1, \ldots, \varphi_m \Rightarrow \psi_1, \ldots, \psi_n.$$

"If all φ_is hold, then some ψ_j holds. "

$$(\varphi_1 \wedge \cdots \wedge \varphi_m) \to (\psi_1 \vee \cdots \vee \psi_n)$$

Sequent Calculus G**K** for Modal Logic **K**

- Axioms:

$$\varphi \Rightarrow \varphi$$

- Weakening rules
- Logical Rules:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} (\Rightarrow \neg) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} (\neg \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Sigma \Rightarrow \Pi}{\varphi \to \psi, \Gamma, \Sigma \Rightarrow \Delta, \Pi} (\to\Rightarrow)$$

- Cut Rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (Cut)$$

- Modal Rule:

$$\frac{\psi_1, \ldots, \psi_n \Rightarrow \varphi}{\Box\psi_1, \ldots, \Box\psi_n \Rightarrow \Box\varphi} (\Box) \quad (n \geqslant 0)$$

Cut Elimination of G**K**

If a sequent is provable in G**K** then it is provable in G**K** w/o (Cut).

(Cor.) Subformula Property of G**K**

If $\Gamma \Rightarrow \Delta$ is provable in G**K** then it is provable in G**K** by a derivation which consists of subformulas of Γ, Δ alone.

Sequent Calculus G**Kt** for Tense Logic

To get G**Kt**, we replace the rule ($\Box$) in G**K** w/:

$$\frac{\blacklozenge\Theta, \Gamma \Rightarrow \varphi}{\Theta, \Box\Gamma \Rightarrow \Box\varphi}\ (\Box_{\mathbf{Kt}}) \qquad \frac{\varphi \Rightarrow \Sigma, \Box\Theta}{\blacklozenge\varphi \Rightarrow \blacklozenge\Sigma, \Theta}\ (\blacklozenge_{\mathbf{Kt}})$$

(Nishimura 1980)

- (*Cut*) is indispensable in G**Kt** as:

$$\dfrac{\dfrac{\Box\neg p \Rightarrow \Box\neg p}{\blacklozenge\Box\neg p \Rightarrow \neg p}\ (\blacklozenge_{\mathbf{Kt}}) \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\ (\neg\Rightarrow)}{p, \blacklozenge\Box\neg p \Rightarrow}\ (Cut)$$

Subformula Property of G**Kt** (Takano 1992)

If $\Gamma \Rightarrow \Delta$ is provable in G**Kt** then it is provable in G**Kt** by a derivation which consists of subformulas of Γ, Δ alone.

Takano's Methods for Subformual Property

- Syntactic Method: Very much like cut-elimination.

Takano (1992) Subformula property as a substitute for cut-elimination in modal propositional logics, Math Jpn, 37(6),1129-1145.

- Semantic Method: Show that system w/ analytic cut is semantically complete.

Takano (2018) A semantical analysis of cut-free calculi for modal logics, Rep. Math. Logic, 54, 43-65.

Seq. Calc. G**CK** for Conditional Logic

To get G**CK**, add the following to the Boolean part of G**K**:

$$\frac{\varphi_0 \Leftrightarrow \cdots \Leftrightarrow \varphi_n \quad \psi_1, \ldots, \psi_n \Rightarrow \psi_0}{[\varphi_1]\psi_1, \ldots, [\varphi_n]\psi_n \Rightarrow [\varphi_0]\psi_0} \; ([\cdot])$$

(Pattinson et al. 2011)

Cut Elimination of GCK** (Pattinson et al. 2011)**

If a sequent is provable in G**CK**
then it is provable in G**CK** w/o (Cut).

Seq. Calc. G**UCK** for Normal Update Logic

To the Boolean part of G**K**, we add the following two:

$$\frac{\varphi_0 \Leftrightarrow \cdots \Leftrightarrow \varphi_n \quad \langle \varphi_1^- \rangle \theta_1, \ldots, \langle \varphi_n^- \rangle \theta_n, \psi_1, \ldots, \psi_n \Rightarrow \psi_0}{\theta_1, \ldots, \theta_n, [\varphi_1]\psi_1, \ldots, [\varphi_n]\psi_n \Rightarrow [\varphi_0]\psi_0} \ ([\cdot]_{\mathbf{UCK}})$$

$$\frac{\varphi_0 \Leftrightarrow \cdots \Leftrightarrow \varphi_n \quad \psi_0 \Rightarrow \psi_1, \ldots, \psi_n, [\varphi_1]\theta_1, \ldots, [\varphi_n]\theta_n}{\langle \varphi_0^- \rangle \psi_0 \Rightarrow \langle \varphi_1^- \rangle \psi_1, \ldots, \langle \varphi_n^- \rangle \psi_n, \theta_1, \ldots, \theta_n} \ (\langle \cdot^- \rangle_{\mathbf{UCK}})$$

Equipollence Result

φ is provable in H**UCK** iff $\Rightarrow \varphi$ is provable in G**UCK**.

- (Cut) is indepensable in G**UCK**:

$$\dfrac{\dfrac{[q]\neg p \Rightarrow [q]\neg p}{\langle q^{-}\rangle[q]\neg p \Rightarrow \neg p}\ (\langle\cdot^{-}\rangle_{\mathbf{UCK}}) \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\ (\neg\Rightarrow)}{p, \langle q^{-}\rangle[q]\neg p \Rightarrow}\ (Cut)$$

$\Gamma \Rightarrow \Delta$ is Ξ-provable in G**UCK** if there is a derivation $\mathcal{D}$ of the sequent such that $\mathcal{D}$ consists of formulas from Ξ.

Main Result of This Talk

TFAE:

1. $\Gamma \Rightarrow \Delta$ is $\mathrm{Sub}(\Gamma, \Delta)$-provable in G**UCK**.
2. $\Gamma \Rightarrow \Delta$ is provable in G**UCK**.
3. $\bigwedge \Gamma \to \bigvee \Delta$ is valid in all finite models.

$\because$ (1) $\Rightarrow$ (2) & (2) $\Rightarrow$ (3) are easy.

We focus on (3) $\Rightarrow$ (1) below.

Corollary

G**UCK** enjoys the subformula property and FMP hence decidability. Therefore, H**UCK** is also decidable.

Proof Outline of (3) $\Rightarrow$ (1)

We prove the contrapositive implication.

1. Suppose: $\Gamma \Rightarrow \Delta$ is not $\mathrm{Sub}(\Gamma, \Delta)$-provable in G**UCK**.
2. Put $\Xi := \mathrm{Sub}(\Gamma, \Delta)$ (finite!).
3. Extend $\Gamma \Rightarrow \Delta$ to a Ξ-complete $\Gamma^+ \Rightarrow \Delta^+$, where "$\Xi$-complete" means:
   - $\Gamma^+ \cup \Delta^+ = \Xi$.
   - $\Gamma^+ \Rightarrow \Delta^+$ is still not Ξ-provable in G**UCK**.
4. Define $M^\Xi = (W^\Xi, (R^\Xi_X)_{X \subseteq W^\Xi}, V^\Xi)$ as:
   - W^Ξ = all Ξ-complete sequents (finite!).
   - R^Ξ_X is defined via $([\cdot]_{\mathbf{UCK}})$ and $(\langle \cdot^- \rangle_{\mathbf{UCK}})$.
   - $\Pi \Rightarrow \Sigma \in V^\Xi(p)$ iff $p \in \Pi$.
5. $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is falsified in the finite M^Ξ. $\Box$

Main Result of This Talk

TFAE:

1. $\Gamma \Rightarrow \Delta$ is $\mathrm{Sub}(\Gamma, \Delta)$-provable in G**UCK**.
2. $\Gamma \Rightarrow \Delta$ is provable in G**UCK**.
3. $\bigwedge \Gamma \to \bigvee \Delta$ is valid in all finite models.

Corollary

G**UCK** enjoys the subformula property and FMP hence decidability. Therefore, H**UCK** is also decidable.

Further Direction

- The paper contains the results on H**UCK** extended w/: (CID) $[\varphi]\varphi$ and/or (CMP) $[\varphi]\psi \rightarrow (\varphi \rightarrow \psi)$.
- Further extension, say w/ (CLEM) $[\varphi]\psi \vee [\varphi]\neg\psi$.
- Syntactic proof of the subformula property of G**UCK**?
- Craig Interpolation Theorem for G**UCK**?

Sequent Calculus G**S5** for Modal Logic **S5**

To get G**S5**, we replace the rule ($\Box$) in G**K** w/:

$$\frac{\Box\Gamma \Rightarrow \Box\Delta, \psi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\psi}\ (\Rightarrow \Box_{\mathbf{S5}}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow)$$

- (Cut) is indispensable in G**S5** as:

$$\dfrac{\dfrac{\dfrac{\neg p, p \Rightarrow}{\Box\neg p, p \Rightarrow}\ (\Box_{\mathbf{S5}} \Rightarrow)}{p \Rightarrow \neg\Box\neg p}\ (\Rightarrow \neg) \qquad \dfrac{\dfrac{\Rightarrow \Box\neg p, \neg\Box\neg p}{\Rightarrow \Box\neg p, \Box\neg\Box\neg p}\ (\Rightarrow \Box_{\mathbf{S5}})}{\neg\Box\neg p \Rightarrow \Box\neg\Box\neg p}\ (\neg \Rightarrow)}{p \Rightarrow \Box\neg\Box\neg p}\ (Cut)$$

Subformula Property of G**S5** (Takano 1992)

If $\Gamma \Rightarrow \Delta$ is provable in G**S5** then it is provable in G**S5** by a derivation which consists of subformulas of Γ, Δ alone.