Satisfaction classes via cut elimination

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Outline

- The objective is to present a fully classical construction of a satisfaction class for the language of first-order arithmetic.
- In the construction, we will be using cut elimination as the main proof technique.
- The main challenge is to modify the cut elimination technique in such a way that it can be applied to a proof system processing possibly non-standard arithmetical formulas.

Classical results

Theorem

A satisfaction class can be constructed in an arbitrary countable recursively saturated model of PA (or a suitable fragment of first-order arithmetic).

- The first proof (for relational arithmetic) is due to Kotlarski, Krajewski and Lachlan (1981).
- Kaye (1991) and Engström (2002) proved the theorem in a setting with function symbols.
- Enayat and Visser (2015) showed how to prove the theorem (for relational arithmetic) by means of classical model-theoretic techniques.

Classical compositional theory of truth

Our setting is that of truth, not satisfaction. Let L_T be obtained from L_{PA} by adding the unary truth predicate '$T(x)$'.

Definition

CT^- is the theory in L_T axiomatized by all the axioms of PA together with the following truth axioms:

- $\forall s, t \in Tm^c \big(T(s = t) \equiv val(s) = val(t) \big)$
- $\forall \varphi \big(Sent_{L_{PA}}(\varphi) \rightarrow (T\neg\varphi \equiv \neg T\varphi) \big)$
- $\forall \varphi \forall \psi \big(Sent_{L_{PA}}(\varphi \vee \psi) \rightarrow (T(\varphi \vee \psi) \equiv (T\varphi \vee T\psi)) \big)$
- $\forall v \forall \varphi(x) \big(Sent_{L_{PA}}(\forall v \varphi(v)) \rightarrow (T(\forall v \varphi(v)) \equiv \forall x T(\varphi(\dot{x}))) \big)$

Main theorem

Theorem

For every countable, recursively saturated model M of PA, there is a set $T \subseteq M$ such that $(M, T) \models CT^-$.

Main stages of the proof:

- We give an external definition of a proof system ML ('M-logic'), which permits us to reason with sentences in the sense of M. The system resembles Gentzen's sequent calculus, but it employs some infinitary rules.
- We demonstrate that if ML is consistent, then it can be extended to a complete set T of M-sentences, which makes all the axioms of CT^- true.
- We show that ML is consistent. This is done by demonstrating that cut elimination holds for ML.

M-logic

All the initial sequents have the form $\varphi \Rightarrow \varphi$ for $\varphi \in Sent_{L_T}$. The following rules of ML are copied directly from Gentzen's system:

- Weakening, left and right (W-left and W-right):

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

- Exchange, left and right (E-left and E-right):

$$\frac{\Gamma, \psi, \varphi, \Gamma' \Rightarrow \Delta}{\Gamma, \varphi, \psi, \Gamma' \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Delta'}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Delta'}$$

- Contraction, left and right (C-left and C-right):

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

- Cut:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Sigma \Rightarrow \Lambda}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}$$

ML rules taken from Gentzen's system

- $\neg$-left and $\neg$-right:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

- $\wedge$-left and $\wedge$-right (for arbitrary sentences A and B such that one of them is φ):

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

- $\vee$-left and $\vee$-right (for arbitrary sentences A and B such that one of them is φ):

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta,}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, A \vee B}$$

- $\rightarrow$-left and $\rightarrow$-right:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Sigma \Rightarrow \Lambda}{\varphi \rightarrow \psi, \Gamma, \Sigma \Rightarrow \Delta, \Lambda} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

Additional rules of ML

In addition, M-logic has the following rules of inference:

- The truth rule for literals (Tr-lit). Let φ be of the form $t = s$ with $M \models t = s$ or of the form $t \neq s$ with $M \models t \neq s$:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

- The M-rule, left and right (M-left, M-right):

$$\frac{\{\varphi(a), \Gamma \Rightarrow \Delta : a \in M\}}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\{\Gamma \Rightarrow \Delta, \varphi(a) : a \in M\}}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)}$$

- $\exists$-right and $\forall$-left:

$$\frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \qquad \frac{\varphi(a), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta}$$

Proofs in ML

- Proofs in ML are (possibly infinite) trees of finite height, where the height of a proof is the length of its maximal path.
- Proofs in ML are purely sentential.
- In *all* the quantifier rules of ML we employ numerals. Thus, for example, in order to apply $\exists$-right, we need a sentence $\varphi(a)$ with a numeral for a.

Lemma

If ML is consistent, then there is a set $T \subseteq M$ such that $(M, T) \models CT^-$

Proof of the lemma (general idea)

Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of the set of $M-$sentences. Given that $T_0 = \emptyset$, we define:

$$T_{n+1} = \begin{cases} T_n \cup \{\varphi_n\} & \text{if ML} \nvdash (T_n \rightarrow \neg\varphi_n) \\ & \text{and } \varphi_n \text{ is not existential,} \\ T_n \cup \{\exists x \psi(x)\} \cup \{\psi(a)\} & \text{if } \varphi_n = \exists x \psi(x) \\ & \text{and ML } \nvdash (T_n \rightarrow \neg\varphi_n), \\ & \text{for an } a \in M \text{ such that} \\ & \text{ML} \nvdash (T_n \rightarrow \neg\psi(a)), \\ T_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Proof of the lemma, continued

Recursive saturation is needed to verify that whenever ML $\nvdash (T_n \to \neg \exists x \psi(x))$, there will exist an $a \in M$ such that ML $\nvdash (T_n \to \neg \psi(a))$.

Let $T = \bigcup_{n \in \omega} T_n$. The proof of the lemma is completed by demonstrating that $(M, T) \models CT^-$ provided that M-logic is consistent.

Consistency of M-logic

Observation

If every sequent provable in M-logic has a cut-free proof, then M-logic is consistent.

Proof.

Take a cut-free proof P of $0 = 1$. Then every sentence in P has to be either atomic or negated atomic. For a sequent S belonging to P, let $h(S)$ (the height of S in P) be defined as the length of maximal path generated by S in d. Let $Tr_0(x)$ be the arithmetical truth predicate for atomic sentences and their negations.
By external induction on the height of sequents in P, it can be demonstrated that for every sequent S in P, if all sentences in the antecedent of S are Tr_0, then some sentence in the succedent of S is Tr_0.
It follows that $M \models Tr_0(0 = 1)$, which is impossible. □

Cut elimination: recapping the classical argument

The aim is to show that the system with the following mix rule admits mix elimination:

$$\frac{\Gamma \Rightarrow \Delta \qquad \Sigma \Rightarrow \Lambda}{\Gamma, \Sigma^* \Rightarrow \Delta^*, \Lambda} \ (\varphi)$$

where Σ and Δ contain φ (the mix formula); Σ^* and Δ^* differ from Σ and Δ only in that they do not contain any occurrence of φ. Since mix and cut produce equivalent proof systems, mix elimination gives us the desired result.

Recapping the classical argument

It is then demonstrated that mix can be eliminated from any proof which contains only a single application of the mix rule in the last step. This is done by double induction on the *degree* of proofs and on the *rank* of proofs. For proofs with mix only in the last step, we define:

- The *left rank* of the proof is the largest number of consecutive sequents in a path starting with the left-hand upper sequent of the mix and such that every sequent in the path contains the mix formula in the succedent.
- The *right rank* of the proof is the largest number of consecutive sequents in a path starting with the right-hand upper sequent of the mix and such that every sequent in the path contains the mix formula in the antecedent.
- The *rank* of the proof is the left rank of the proof + the right rank of the proof.
- The *degree* of the proof is the syntactic complexity of the mix formula.

Main problem

- There is no problem in our setting with induction on the rank of proofs, since both the left and the right rank of the proof in ML will always be a (standard) natural number, restricted by the height of the proof.
- Induction on the degree of proofs is problematic. Since the mix formula might be non-standard, its syntactic complexity might be a non-standard element of M. Arguing externally by induction on non-standard numbers is clearly an invalid move and this is the main obstacle complicating the situation.

The remedy (intuition)

The remedy is to replace the general notion of a degree with a notion relativized to a proof.

Assume that we are given a proof *P* with mix only in the last step, that eliminates the mix formula φ. The guiding intuition is that in the cut elimination proof the syntactic shape of φ matters only comparatively.

For example, φ might have the form $\neg\psi$. The intuition is that this will matter only provided that ψ itself (without negation) appears somewhere in *P*; otherwise φ might just as well be treated as a formula of complexity 0, even if it is non-standard.

Basic definitions

Definition

- $x \lhd y$ ('x is a direct subsentence of y') is an abbreviation of the following arithmetical formula:

$$Sent_{L_{PA}}(x) \wedge Sent_{L_{PA}}(y) \wedge \Big(\exists \psi \in Sent_{L_{PA}}(y = \ulcorner \neg \psi \urcorner \wedge x = \psi) \vee \exists \varphi, \psi \in Sent_{L_{PA}}(y = \ulcorner \varphi \circ \psi \urcorner \wedge x = \varphi \vee x = \varphi) \vee \exists \theta(x) \in Fm_{L_{PA}} \exists a \, \exists v \in Var(y = \ulcorner Qv\theta(v) \urcorner \wedge x = \ulcorner \theta(a) \urcorner)\Big).$$

- Let $\varphi \in Sent_{L_{PA}}(M)$. We say that s is a $\lhd$-sequence for φ iff $s_0 = \varphi$ and for every $k < lh(s) - 1$ $s_{k+1} \lhd s_k$.

The notion of a degree

The notion of a degree is defined in the following way.

Definition

Let P be an arbitrary proof in ML with mix used only in the last step. Let φ be the mix formula in P.

- $d(\varphi, P)$ (the degree of φ in P) $= sup\{lh(s) : s$ is a $\triangleleft$-sequence for φ such that for every $k < lh(s)$ $s_k \in P\}$.
- $d(P)$ (the degree of P) is the same as $d(\varphi, P)$.

Lemma

Let P be an arbitrary proof in ML *with mix used only in the last step. Then $d(P)$ is a natural number (in other words, it is never ω).*

Cut elimination: general setting

Given the lemma, we demonstrate that mix can be eliminated from any proof which contains only a single application of the mix rule in the last step.

Let us assume (main induction) that cut can be eliminated in every proof of a degree $< n$. Let us also assume (subinduction) that cut can be eliminated in every proof of a degree n but with rank $< k$. Our task is to show that cut can be eliminated in proofs of degree n and rank k.

The proof starts with the case of $k = 2$ (the lowest possible rank) and proceeds by analysing subcases.

Cut elimination: chosen cases

Let us assume that the mix formula is $\forall x \varphi(x)$. Then the last stage of the proof runs as follows:

$$\dfrac{M\text{-right}\ \dfrac{\{\Gamma \Rightarrow \Delta, \varphi(a) : a \in M\}}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \qquad \dfrac{\varphi(c), \Sigma \Rightarrow \Lambda}{\forall x \varphi(x), \Sigma \Rightarrow \Lambda}\ \forall\text{-left}}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}\ \text{mix}$$

We can then eliminate mix in the following way:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi(c) \qquad \varphi(c), \Sigma \Rightarrow \Lambda}{\Gamma, \Sigma^{*} \Rightarrow \Delta^{*}, \Lambda}\ \text{mix}}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}\ \text{possibly, some weakenings and exchanges}$$

We use the inductive assumption here, namely, we show that the same end sequent can be obtained by applying mix to the formula $\varphi(c)$, which has the degree $n-1$ in P (the sentence $\forall x \varphi(x)$ has the degree n). Observe that in the modified proof $\varphi(c)$ will have the degree not larger than $n-1$.

Cut elimination: chosen cases

When $k > 2$, we have in addition the case of (Tr-Lit) to analyse. Thus, the last stage of the proof might run as follows:

$$\dfrac{\text{Tr-Lit}\,\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \Sigma \Rightarrow \Lambda}{\Gamma, \Sigma^* \Rightarrow \Delta^*, \Lambda}\,\text{mix}$$

Then we eliminate mix in the following way:

$$\dfrac{\text{mix}\,\dfrac{\varphi, \Gamma \Rightarrow \Delta \qquad \Sigma \Rightarrow \Lambda}{\varphi, \Gamma, \Sigma^* \Rightarrow \Delta^*, \Lambda}}{\Gamma, \Sigma^* \Rightarrow \Delta^*, \Lambda}\,\text{Tr-Lit}$$

Since the new proof has lower rank than *k* (we moved the mix up the derivation), the inductive hypothesis applies and the mix rule is eliminable. The case of (Tr-Lit) being used to obtain the right-hand upper sequent of the mix is very similar.

THE END

Thanks for your attention!!!