Model Theory for Sheaves of Modules

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For instance solution sets of systems of equations: if M is a field and $\varphi(\overline{x})$ is a system of equations $\bigwedge_{i=1}^{m} p_i(\overline{x}) = 0$ where p_i is a polynomial in $x_1, \ldots, x_n$ with coefficients in M, then the solution set $\varphi(M)$ is a typical affine variety (a subvariety of affine n-space over M).

Modules: that is, **representations** of mathematical structures (for example a group G) by actions on simple mathematical structures such as abelian groups or vector spaces V. So a module is given by a homomorphism (for example of groups) from the structure to the endomorphism algebra of the representing space ($\rho : G \to \mathrm{End}(V)$).

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Usually the structure being represented may be taken to be a ring R, for example the ring of integers $\mathbb{Z}$, or a ring of matrices $M_n(K)$, or a polynomial ring $K[T_1, \dots, T_n]$.

Model theory for (*R*-)modules: given a ring R, we set up a language $\mathcal{L}_R$ with a constant symbol 0 and binary operation symbol $+$ with which to express the underlying abelian group structure of a module, and, for each $r \in R$, a 1-ary function symbol with which to express (scalar) multiplication by r.

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A typical atomic formula is, modulo the theory of, say right, R-modules, of the form $\sum_{i=1}^{n} x_i r_i = 0$ with the $r_i \in R$ and variables x_i. We build the (finitary, classical) language $\mathcal{L}_R$ from these atomic formulas in the usual way.

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That is, a formula of the form $\exists y_1, \ldots, y_k \left(\sum_{i=1}^n x_i r_i + \sum_{j=1}^k y_j s_j = 0 \right)$ - a typical **pp** (for "positive primitive", also called **regular**) formula - is equivalent to one of the form $\sum_{i=1}^n x_i t_i = 0$. But, over general rings, we must keep the existential quantification, so the complexity of formulas is higher. We do, however, have the following:

pp-elimination of quantifiers: Every formula for R-modules is, modulo the theory of R-modules, equivalent to a finite boolean combination of invariants sentences and pp formulas.
An "invariants sentence" is a sentence of $\mathcal{L}_R$ expressing that the index of the solution set of a pp formula $\psi(\overline{x})$ in that of some other pp formula $\varphi(\overline{x})$ is at least N, for some natural number N. This makes sense because solution sets of pp formulas are abelian groups.

pp formulas and algebra: Thus the pp formulas are the most important for the model theory of modules. They are also exactly those whose solution sets are preserved by homomorphisms (if $f : M \to N$ is a homomorphism of R-modules and φ a pp formula, then $f\varphi(M) \subseteq \varphi(N)$).
This is reflected in a strong connection between model theory and algebra for modules, with the former having many applications to the latter.

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In particular, given X, a **ringed space** $\mathcal{R}$ over X is given by the data:
for each open set $U \subseteq X$, a ring RU (with 1);
for each inclusion $V \subset U$ of open sets, a ring homomorphism (**restriction**)
$r_{UV} : RU \to RV$, such that $r_{UU} = 1_{RU}$ and, if $W \subseteq V \subseteq U$, then $r_{VW} r_{UV} = r_{UW}$.

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given, for each λ, some $s_\lambda \in RU_\lambda$, such that, for every λ, μ, $r_{U_\lambda U_\lambda \cap U_\mu}(s_\lambda) = r_{U_\mu U_\lambda \cap U_\mu}(s_\mu)$, there is $s \in RU$ such that, for every λ, $r_{UU_\lambda}(s) = s_\lambda$.

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For instance, X might be a manifold and RU the ring of continuous functions from U to $\mathbb{R}$.

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The category $\mathrm{Mod}\text{-}\mathcal{R}$ **of** $\mathcal{R}$**-modules**: This is a Grothendieck abelian category. If X, together with $\mathcal{R}$, is an algebraic variety, then we also have the full subcategory $\mathrm{Qcoh}(X)$ of quasicoherent sheaves over X, which are those which look locally (on each member of an affine cover) like a module made into a sheaf by localisation.

Model theory for sheaves of modules: At first sight it is not at all obvious that we can treat sheaves of modules model-theoretically in the same way as ordinary modules (that, note, is the case where X is a 1-point space). But, with a change in viewpoint, we can.

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Then we can regard the elements of V_i as being the elements of V of sort i.

Languages for multisorted structures: these are as usual (1-sorted) first order languages but:

variables are sorted;
symbols for constants are sorted;
each function symbol has domain a finite product of sorts and codomain a sort;
each relation symbol is on a finite product of sorts.

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In terms of its representations, a multisorted ring $\mathbf{R}$ is equivalent to a small preadditive category, the left modules over $\mathbf{R}$ being exactly the additive functors from $\mathbf{R}$, regarded as a preadditive category, to the category $\mathbf{Ab}$ of abelian groups (the right modules are the contravariant functors).

Multisorted Modules: As just stated, these can be regarded as additive functors, but we can take a more standard, set-theoretic, view of them, essentially replacing a functor by its image. In this view, a multisorted module consists of a set of abelian groups (indexed by a fixed set I, which will also index the sorts of the corresponding language) and a set of additive maps between these groups.

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A many-sorted=multisorted module can be thought of as a representation of a quiver. *At* each vertex/sort there is a ring acting but there are also (multiplication-by-)scalar actions *between* sorts.

Languages for multisorted modules:
for each sort σ we have symbols 0_σ, $+_\sigma$ with which to express the abelian group structure;
for each element r of the multisorted ring, with domain σ and codomain τ, we have a 1-ary, corespondingly sorted, function symbol in the language (covariant for left modules, contravariant for right modules).

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Therefore, although this language would be suitable for presheaves, the class of sheaves would not necessarily be an axiomatisable subclass of the class of presheaves.

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For instance, if X is a **noetherian** space (has the descending chain condition on closed subsets), then every open set is compact. The ringed spaces arising in practice, in particular in algebraic geometry, very often have this property.

Finitely accessible categories: If X is a topological space satisfying the condition that there is a basis, closed under intersection, of compact open sets, and if $\mathcal{R}$ is any sheaf of rings over X, then the category of $\mathcal{R}$-modules is **finitely accessible**, indeed locally finitely presented.

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In that case, the structures in the category are amenable to analysis using multisorted first order classical logic. Therefore, in the case of sheaves over such a ringed space all the usual methods and theorems of model theory for ordinary, 1-sorted, modules, apply.

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