A Modal Aleatoric Calculus for Probabilistic Reasoning

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Automated reasoning for game playing agents

Many artificial intelligence applications require agents to reason with uncertain information. In games, this could be the shuffle of a deck of cards, or the amount of resources another player has. Furthermore, game playing agents often require mixed strategies for optimal play, where actions are chosen probabilistically, so as not to sign-post information to opposition players. There is a need for probabilistic reasoning tools for game playing agents.
Aleatoric Reasoning

Aleatoric comes from the Latin word *aleator* meaning dice player, and literally means “by the throw of dice”. We model agents as aleators:

- The probability of another player holds the ace could be calculated as 1/3 using a simple card counting argument.
- If the other player was aggressive, they would have likely (9/10) already played the ace.
- From what we have observed of the other player, we may give a 3/5 chance of them being aggressive.

All of this data and assumptions can be compiled to give a likelihood of the other player holding the ace. This is aleatoric reasoning.
Aleatoric variables and betting protocols

We are interested in building a formal reasoning system for AI agents, using the notion of independent random events rather than Boolean propositions. An aleatoric variable can be thought of as a (biased) coin, $x$, labelled with $\top$ and $\bot$. If it lands $\top$ we interpret it as “yes”, and if it lands $\bot$ we interpret it as “no”.

Complex propositions can be formed as betting protocols: “If $x$ lands $\top$ flip $y$ and return $y$’s answer, otherwise return $\bot$”, or “If $x$ lands $\top$ three times in a row, flip $y$ three time and return the majority answer”
Aleatoric concepts

We can use these aleatoric variable and betting protocols to define *concepts*. Consider the concept of arriving at work *Dry*. For this to happen, it could either be that it was raining (2/3), but we had an umbrella (1/2), or it wasn’t raining (1/3), and the sprinklers didn’t come one (1/3).

We may associate probabilities with each of these events to compute the likelihood of arrive at work dry (4/9).
The aleatoric calculus: syntax and semantics

The *aleatory calculus* (AC) is a generalisation of propositional logic that treats all formulas as independent random variables.

\[
\phi ::= \top | \bot | x | (\phi?\phi:\phi)
\]

where \( x \in X \).

- \( \top \) is always true, and \( \bot \) is never true.
- \( x \) is an independent random variable, that is sampled every time it appears.
- \( (\alpha?\beta:\gamma) \) is the *if-then-else* operator.

The semantics are given by a function \( f : X \rightarrow [0,1] \), and the probability of a formula holding, given \( f \) is:

\[
P^f(\top) = 1 \quad P^f(\bot) = 0 \quad P^f(x) = f(x)
\]

and

\[
P^f((\alpha?\beta:\gamma)) = P^f(\alpha)P^f(\beta) + (1 - P^f(\alpha))P^f(\gamma).
\]
Abbreviations

Some abbreviations we can define in AC are as follows:

\[ \alpha \land \beta = (\alpha \land \beta : \bot) \]
\[ \alpha \lor \beta = (\alpha \lor \beta : \top) \]
\[ \alpha \rightarrow \beta = (\alpha \rightarrow \beta : \top) \]
\[ \neg \alpha = (\alpha \rightarrow \bot : \top) \]
\[ \alpha \leftrightarrow \beta = (\alpha \land \beta : \neg \beta) \]
\[ \alpha^0/b = \top \]
\[ \alpha^a/b = \bot \text{ if } b < a \]
\[ \alpha^a/b = (\alpha \land \alpha^{(a-1)/(b-1)} : \alpha^{a/(b-1)}) \text{ if } a \leq b \]

where \( a \) and \( b \) are natural numbers. The formula \( \alpha^{a/b} \) is the expectation of \( \alpha \) being sampled \( a \) out of \( b \) times.
The calculus

The equivalences of the base aleatory calculus can be found by applying uniform substitutions of the following axioms:

- **id** \( x \equiv x \)
- **ignore** \( (x?y:y) \equiv y \)
- **always** \( (\top?x:y) \equiv x \)
- **never** \( (\bot?x:y) \equiv y \)
- **vacuous** \( (x?\top:\bot) \equiv x \)
- **tree** \( ((x?y:z)?p:q) \equiv (x?(y?p:q):(z?p:q)) \)
- **swap** \( (x?(y?p:q):(y?r:s)) \equiv (y?(x?p:r):(x?q:s)) \)

The first five are reasonably obvious. We will look at the next two in detail.
The Tree rule

The tree rule is: \(((x?y:z)?p:q) \simeq (x?(y?p:q):(z?p:q))\)

\[
P(((x?y:z)?p:q)) \\
= P(x)P(y)P(p) + P(\bar{x})P(z)P(p) + P(x)P(\bar{y})P(q) + P(\bar{x})P(\bar{z})P(q) \\
= P(x)(P(y)P(p) + P(\bar{y})P(q)) + P(\bar{x})(P(z)P(p) + P(\bar{z})P(q)) \\
= P(x)P((y?p:q)) + P(\bar{x})P((z?p:q)) \\
= P((x?(y?p:q):(z?p:q)))
\]
The **Swap** rule

The swap rule is: \((x?(y?p:q):(y?r:s)) \simeq (y?(x?p:r):(x?q:s))\)

\[
P((x?(y?z:p):(y?r:z))) = P(x)P(y)P(z) + P(x)P(\overline{y})P(p) + P(\overline{x})P(y)P(r) + P(\overline{x})P(\overline{y})P(z) \\
= P(y)(P(x)P(z) + P(\overline{x})P(r)) + P(\overline{y})(P(x)P(p) + P(\overline{x})P(z)) \\
= P((y?(x?z:r):(x?p:z)))
\]
Completeness

**Theorem:** For any pair of semantically equivalent aleatoric calculus formulae $\phi$ and $\psi$, we can show $\phi \simeq \psi$.

To do this we:

- shown that every formula can be converted into *tree-form* using the rule **Tree**.
- we then show that every *tree-form* formula can be converted into some standard form, using **Swap** to reorder the tree.
- finally we show that any two semantically equivalent formulas will have the same standard form.
Dependence and possible worlds

In the rain and sprinkle example our model had the likelihood of us having an umbrella independent to the likelihood of it raining. The probability of arrive to work dry is better expressed as the probability of us having an umbrella, *given it rains*, and the probability of the sprinklers being on.

That is we can think of a distribution of possible worlds, and in each world there is a likelihood of it raining, a likelihood of us having an umbrella, and a likelihood of the sprinklers being on, and these are dependent on the world.

\[
\begin{align*}
\text{winter} & : r : \frac{2}{3} \\
& : u : \frac{3}{4} \\
& : s : \frac{1}{10}
\end{align*}
\]

\[
\begin{align*}
\text{summer} & : r : \frac{1}{4} \\
& : u : \frac{1}{10} \\
& : s : \frac{1}{2}
\end{align*}
\]
Probability models

A probability model is specified by the tuple $P = (W, \pi, f)$, where:

- $W$ is a set of possible worlds.
- $\pi : N \rightarrow W \rightarrow PD(W)$ assigns for each world $w \in W$, a probability distribution $\pi_i(w)$ over $W$.
- $f : W \rightarrow X \rightarrow [0, 1]$ is a probability assignment so $f_w(x)$ is the probability of $x$ being true at $w$.

A probability model for an aleator who does not know whether the die is four sided ($w_4$) or six sided ($w_6$).
The marginal operator

\[ P_w((\alpha | \beta)_i) = \frac{E_w^i(\alpha \land \beta)}{E_w^i(\beta)} \]

where \( E_w^i(\alpha) = \sum_{u \in W} \pi_i(w, u).P_u(\alpha) \).

Some abbreviations we can define in the modal aleatoric calculus are as follows:

- \( E_i\alpha = (\alpha | \top)_i \)
- \( B_i\alpha = (\bot | \neg \alpha)_i \)

- \( E_i\alpha \) is \( i \)'s expectation of \( \alpha \) being sampled.
- \( B_i\alpha \) is true if agent \( i \) assigns 0 chance of \( \alpha \) not being sampled.
Example:

Suppose in our current world \( w \), we do not know whether it is Summer, Fall or Winter, and we want to calculate the likelihood of having an umbrella, given it rains.

\[
(U | R)_i = \frac{3}{10} + \frac{3}{40} + \frac{1}{40} = 0.4
\]
The modal aleatoric calculus

The modal aleatoric calculus augments the rules of the aleatoric calculus with the following equivalences.

\[ A_0 : \quad ((x?z:w)|y)_i \simeq ((x|y)_i?(z|(x?y:\bot))_i:(w|(x\bot:y))_i). \]

\[ A_1 : \quad (\bot|x)_i \land (x|y)_i \simeq (\bot|x \lor y)_i \]

\[ A_2 : \quad (\bot|x)_i \not\leftrightarrow ((\bot|x)_i?(\bot|x)_i:\neg(\bot|x)_i) \]

\[ A_3 : \quad (\top|x)_i \not\leftrightarrow \top \]

\[ A_4 : \quad (x|\bot)_i \not\leftrightarrow \top \]

The main axiom is the axiom \( A_0 \) which is a rough analogue of the \( K \) axiom in modal logic. If we substitute \( \top \) for \( y \) and \( \bot \) for \( w \), we have:

\[ E_i x \land (y|x)_i \simeq E_i (x \land y) \]

All these axioms can be shown to be sound by algebraic reasoning.
Results

- The axiomatisation for base aleatory calculus is complete.
- The axiomatisation for the full modal aleatory calculus is sound.
- We conjecture the axiomatization for the modal aleatory calculus is complete.
- The aleatory calculus is a generalisation of modal logic.
- The aleatory calculus has polynomial time model checking.
Related work

There have been numerous probabilistic extensions to DEL: Halpern and Grunwald (2003), Kooi (2003), Aucher (2005), Baltag and Smets (2008), van Benthem, Gerbrandy and Kooi (2008), Sack (2009), ...

**DEFINITION 2 (Language of PDEL).** Let a countable set of propositional variables $\mathcal{P}$ and a finite set of agents $\mathcal{A}$ be given. The language of PDEL $\mathcal{L}_{\mathcal{P}, \mathcal{A}}$ is given by the following rule in extended Backus–Naur form:

$$\varphi ::= p \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \Box_a \varphi \mid [\varphi_1] \varphi_2 \mid q_1 P_a(\varphi_1) + \cdots + q_n P_a(\varphi_n) \geq q,$$

Additionally, Kozen (1985) has formalised reasoning about programs with random variable, Pearl (1994) has given a calculus of probabilistic actions, and Wild et al (2018) have investigated fuzzy modal logic.
Future work

In future work we will:

▶ aim to show the calculus is complete;
▶ consider a more general set informative events, such as private announcements, and action models; and
▶ generalise these results to other modal settings such as description logic, temporal logics, and continuous domains.
▶ develop an aleatoric model theory.
Questions?

Theory of Games: the Finger Game called Morra

The game is played between two players and, in its simplest version, goes as follows. The two players move simultaneously. Each shows either one or two fingers, and at the same time guesses whether his opponent is showing one or two fingers. If both players guess right, or both guess wrong, no one pays. If only one player guesses right, he wins from the other as many chips as the two players together showed fingers.

Thus each player has the choice of four courses:

(a) 1  (b) 2  (c) 1  (d) 2

If his call is right and his opponent's wrong, course (a) will win two chips, course (b) and (c) will win three chips, and course (d) will win four chips.

The game is fair, but a player who knows the right strategy will (with average luck) win against one who does not. The right strategy is to ignore courses (a) and (d), and to play courses (b) and (c) in the ratio of 7:5. That is, the right strategy is, in any 12 calls,

2 on the average 1 on the average

This strategy is unlikely to be guessed by a gambler who plays hunches.
Example: Pig

A simple version of the game *pig* uses a four sided die, and players take turns. Each turn, the player rolls the dice as many times as they like, adding the numbers the roll to their turn total. However, if they roll a 1, their turn total is set to 0, and their turn ends. They can elect to stop at any time, in which case their turn total is added to their score. A simple two world representation of the game pig, where the dice is possibly biased.
Example: reasoning in pig

We can now build aleatoric formula describing various situations, assuming the dice is actually biased, assuming the actual world is the biased world.

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>Description</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>bust</td>
<td>(gt2?⊥:odd)</td>
<td>prob of rolling 1</td>
<td>0.15</td>
</tr>
<tr>
<td>four</td>
<td>(odd?⊥:gt2)</td>
<td>prob of rolling 4</td>
<td>0.35</td>
</tr>
<tr>
<td>thinkBust</td>
<td>(bust</td>
<td>⊤)</td>
<td>chance of rolling 1</td>
</tr>
<tr>
<td>think-4-1</td>
<td>(bust</td>
<td>four)</td>
<td>chance of 1 given 4</td>
</tr>
<tr>
<td>rollAgain</td>
<td>(thinkBust$^{1/2}$?risk:⊤)</td>
<td>roll again risk</td>
<td>0.77</td>
</tr>
</tbody>
</table>
The Resistance is a game based on bluffing, deduction and strategy.

The basic premise is that every player is given a role: either a true resistance member or a spy for the government.

The spies (about a third of the players) know who the other spies are, but the resistance members do not.

The players go on a series of missions, which the spies seek to sabotage, without revealing their identity.

Everybody gets to vote on who goes on a mission.
The resistance common prior. The worlds are labelled with the agents who are spies, and each agent considers all linked worlds equally likely.
Agent 2 and 3 go on a mission, and 2 betrays

The probability model after one of agents 2 and 3 betrays the mission.