The Finite Embeddability Property for Topological Quasi-Boolean Algebra 5

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2019/03/03

Quasi-Boolean algebra

Definition (qBa)

A *quasi-Boolean algebra* (qBa) is an algebra $\mathbf{A} = (\mathrm{A}, \wedge, \vee, \neg, 0, 1)$ where $(\mathrm{A}, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and $\neg$ is an unary operation on *A* such that the following conditions hold for all $a, b \in \mathrm{A}$:

$$(\mathrm{DN}) \quad \neg\neg a = a, \quad (\mathrm{DM}) \quad \neg(a \vee b) = \neg a \wedge \neg b$$

Topological quasi-Boolean algebra

Definition (tqBa)

A *topological quasi-Boolean algebra* (tqBa) is an algebra $\mathbf{A} = (\mathrm{A}, \wedge, \vee, \neg, 0, 1, \Box)$ where $(\mathrm{A}, \wedge, \vee, \neg, 0, 1)$ is a quasi-Boolean algebra, and $\Box$ is an unary operation on A such that for all $a, b \in \mathrm{A}$:

$$(\mathrm{K}_\Box) \quad \Box(a \wedge b) = \Box a \wedge \Box b, \quad (\mathrm{N}_\Box) \quad \Box\top = \top$$

$$(\mathrm{T}_\Box) \quad \Box a \leq a, \quad (4_\Box) \quad \Box a \leq \Box\Box a$$

Topological quasi-Boolean algebra 5

Definition (tqBa5)

A *topological quasi-Boolean algebra 5* is a topological quasi-Boolean algebra $\mathbf{A} = (\mathrm{A}, \wedge, \vee, \neg, \Box, 0, 1)$ such that for all $a \in \mathrm{A}$:

$$(5) \quad \Diamond a \leq \Box\Diamond a,$$

where $\Diamond$ is an unary operation on A defined by $\Diamond a := \neg\Box\neg a$.

(1) Banerjee, M., Chakraborty, M.: Rough algebra. Bulletin of Polish Academy of Sciences (Mathematics) **41**(4), 293–297 (1993)

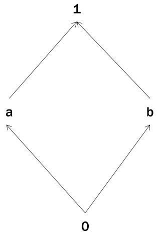
(2) Banerjee, M., Chakraborty, M.: Rough sets through algebraic logic. Fundamenta Informaticae **28**(3-4), 211–221 (1996)

Example (Banerjee, M.)

Let us consider the lattice whose Hasse diagram is shown in Fig.2 and $\neg$, $\Box$ are defined as follows:

	1	0	a	b
$\neg$	0	1	a	b

	1	0	a	b
$\Box$	1	0	a	b



Finite embeddability property

An embedding from a partial algebra **B** into an algebra **C**, we mean an injection $h : \mathrm{B} \mapsto \mathrm{C}$ such that if $b_1, \ldots, b_n, f^{\mathbf{B}}(b_1, \ldots, b_n) \in \mathrm{B}$, then

$$h(f^{\mathbf{B}}(b_1, \ldots, b_n)) = f^{\mathbf{C}}(h(b_1), \ldots, h(b_n)).$$

If **B** and **C** are ordered, then h is required to be an order embedding i.e. $a \leq^{\mathbf{B}} b \Leftrightarrow h(a) \leq^{\mathbf{C}} h(b)$.

Definition (FEP)

A class $\mathbb{K}$ of algebras has *the finite embeddability property* (FEP), if every finite partial subalgebra of a member of $\mathbb{K}$ can be embedded into a finite member of $\mathbb{K}$.

Variety

Lemma (Birkhoff)

A class $\mathbb{K}$ of algebras is a variety if and only if $\mathbb{K}$ is an equational class

(3) G. Birkhoff, On the structure of abstract algebras 31 (1935), 433-454.

Lemma

The class of tqBa5 is a variety.

(4) Saha, A., Sen, J., Chakraborty, M.: Algebraic structures in the vicinity of pre-rough algebra and their logics. Information Science 282, 296C320 (2014).

(2) Banerjee, M., Chakraborty, M.: Rough sets through algebraic logic. Fundamenta Informati- cae 28(3-4), 211C221 (1996).

Strong Finite Model Property

Definition (SFMP)

The strong finite model property (SFMP) i.e. every quasi-equation (quasi-identity) which fails to hold in a class $\mathbb{K}$ of algebras can be falsified in a finite member of $\mathbb{K}$.

FEP and SFMP

Lemma ((5)—Lemma 6.40)

For variety $\mathbb{K}$ of finite type the following are equivalent:

(1) $\mathbb{K}$ *has FEP*

(2) $\mathbb{K}$ *have SFMP*

(5) Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Springer (2007).

- If a formal system $\mathbb{S}$ is strongly complete with respect to a class $\mathbb{K}$ of algebras SFMP for $\mathbb{S}$ with respect to $\mathbb{K}$ yields SFMP for $\mathbb{K}$.

Sequent system for tqBa5 I

Definition (Formula)

The language of the logic of tqBa5 is defined as follows

$$\alpha ::= p \mid \bot \mid \top \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \neg\alpha \mid \Diamond\alpha \mid \Box\alpha,$$

where $p \in$ **Prop**, the set of propositional variables.

Definition (Formula structures)

Formula structure are defined as follows with a unary structural operation $\langle\rangle$:

- α is a formula structure if α is a formula
- $\langle\Gamma\rangle^i$ is a formula structure if Γ is a formula structure

Definition (Sequent)

Sequent is an expression of the form $\langle\alpha\rangle^i \Rightarrow \beta$ where $i \geq 0$ for some formulae α and β.

Sequent system for tqBa5 II

The Gentzen sequent calculus G5 consists of the following axioms and inference rules:

(1) Axioms:

$$\text{(Id)}\ \varphi \Rightarrow \varphi \qquad (\bot)\ \langle\bot\rangle^i \Rightarrow \varphi \qquad (\top)\ \langle\varphi\rangle^i \Rightarrow \top$$

$$\text{(D)}\ \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \qquad \text{(DN)}\ \varphi \Leftrightarrow \neg\neg\varphi$$

(2) Connective rules:

$$\frac{\langle\varphi\rangle^i \Rightarrow \chi}{\langle\varphi \wedge \psi\rangle^i \Rightarrow \chi}(\wedge L) \quad \frac{\langle\chi\rangle^i \Rightarrow \varphi \quad \langle\chi\rangle^i \Rightarrow \psi}{\langle\chi\rangle^i \Rightarrow \varphi \wedge \psi}(\wedge \mathrm{R})$$

$$\frac{\langle\varphi\rangle^i \Rightarrow \chi \quad \langle\psi\rangle^i \Rightarrow \chi}{\langle\varphi \vee \psi\rangle^i \Rightarrow \chi}(\vee \mathrm{L}) \quad \frac{\langle\chi\rangle^i \Rightarrow \psi}{\langle\chi\rangle^i \Rightarrow \psi \vee \varphi}(\vee \mathrm{R})$$

(3) Modal rules

$$\frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\Diamond\varphi\rangle^{i} \Rightarrow \psi}(\Diamond\mathrm{L}) \quad \frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\varphi\rangle^{i+1} \Rightarrow \Diamond\psi}(\Diamond\mathrm{R})$$

$$\frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\Box\varphi\rangle^{i+1} \Rightarrow \psi}(\Box\mathrm{L}) \quad \frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i} \Rightarrow \Box\psi}(\Box\mathrm{R})$$

$$\frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i} \Rightarrow \psi}(\mathrm{T}) \quad \frac{\langle\varphi\rangle^{i+1} \Rightarrow \psi}{\langle\varphi\rangle^{i+2} \Rightarrow \psi}(4) \quad \frac{\langle\varphi\rangle^{i} \Rightarrow \psi}{\langle\neg\psi\rangle^{i} \Rightarrow \neg\varphi}(\Diamond\Box)$$

(4) Cut rule

$$\frac{\langle\varphi\rangle^{i} \Rightarrow \chi \quad \langle\chi\rangle^{j} \Rightarrow \psi}{\langle\varphi\rangle^{i+j} \Rightarrow \psi}(\mathrm{Cut})$$

Algebraic models

An algebraic model of G5 is a pair $(\mathbf{G}, \sigma)$ such that $\mathbf{G}$ is a $\mathrm{tqBa5}$, and σ is a mapping from Prop into G, called a *valuation*, which is extended to formulae and formula trees as follows:

$$\sigma(\Box\alpha) = \Box\sigma(\alpha), \sigma(\Diamond\alpha) = \Diamond\sigma(\alpha)$$

$$\sigma(\alpha \wedge \beta) = \sigma(\alpha) \wedge \sigma(\beta), \quad \sigma(\alpha \vee \beta) = \sigma(\alpha) \vee \sigma(\beta),$$
$$\sigma(\neg\alpha) = \neg\sigma(\alpha), \quad \sigma(\langle\alpha\rangle^{i+1}) = \Diamond\sigma(\langle\alpha\rangle^{i}).$$

Definition (True)

$(\mathbf{G}, \sigma) \models \langle\alpha\rangle^i \Rightarrow \beta$, if $\sigma(\langle\alpha\rangle^i) \leq \sigma(\beta)$ (here $\leq$ is the lattice order in $\mathbf{G}$).

$\Phi \models \langle\alpha\rangle^i \Rightarrow \beta$ with respect to $\mathrm{tqBa5s}$, if $\mathbf{G}, \sigma \vdash \langle\alpha\rangle^i \Rightarrow \beta$ in all models $(\mathbf{G}, \sigma)$ such that $\mathbb{G} \in \mathrm{tqBa5}$ and for any sequent $\langle\varphi\rangle^j \Rightarrow \psi \in \Phi$, $\mathbf{G}, \sigma \vdash \langle\varphi\rangle^j \Rightarrow \psi$

Strongly complete and SFMP

Theorem (Strongly completeness)

G5 is strongly complete with respect to tqBa5s: for any set of sequents Φ *and any sequent* $\langle\alpha\rangle^i \Rightarrow \beta$ *,* $\Phi \vdash_{\mathrm{G5}} \langle\alpha\rangle^i \Rightarrow \beta$ *if and only if* $\Phi \models \langle\alpha\rangle^i \Rightarrow \beta$ *with respect to tqBa5.*

Definition (SFMP)

For any finite set of sequents Φ, if $\Phi \nvdash_{\mathrm{G5}} \langle\alpha\rangle^i \Rightarrow \beta$, then there exists a finite $\mathbf{G} \in \mathrm{tqBa5s}$ and a valuation σ such that all sequents from Φ are true in $(\mathbf{G}, \sigma)$, but $\langle\alpha\rangle^i \Rightarrow \beta$ is not.

Interpolant I

Definition (T-sequent)

A sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is called a T sequent if $\alpha, \beta \in T$.

Definition (T-derivation)

A derivation from Φ in G5 of a T-sequent $\langle\alpha\rangle^i \Rightarrow \beta$ is called a T-derivation if all sequents appearing in the derivation are T-sequents, which is denoted by $\Phi \vdash_{G5} \langle\alpha\rangle^i \Rightarrow_T \beta$.

Interpolant II

Definition (Interpolant)

Assume that $\Phi \vdash_{G5} \langle\varphi\rangle^{i+j} \Rightarrow_T \psi$. A formula γ is called a T interpolant of $\langle\varphi\rangle^i$ if $\gamma \in T$, $\Phi \vdash_{G5} \langle\varphi\rangle^i \Rightarrow_T \gamma$ and $\Phi \vdash_{G5} \langle\gamma\rangle^j \Rightarrow_T \psi$ and additionally $\Phi \vdash_{G5} \langle\gamma\rangle \Rightarrow_T \gamma$ if $i \geq 1$.

Lemma (Interpolant)

If $\Phi \vdash_{G5} \langle\varphi\rangle^{i+j} \Rightarrow_T \psi$, then $\langle\varphi\rangle^i$ has a T interpolant.

- T is closed under $\neg, \wedge, \vee$ and subformulas.

Definitions

Definition ($\leq_T$)

Let $\langle\alpha_1\rangle^i, \langle\alpha_2\rangle^j \in T^s$, we say $\langle\alpha_1\rangle^i \leq_T \langle\alpha_2\rangle^j$ if $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha_2\rangle^j\rangle^t \Rightarrow_T \beta$ implies $\Phi \vdash_{\mathrm{G5}} \langle\langle\alpha_1\rangle^i\rangle^t \Rightarrow_T \beta$ for any context $\langle\rangle^t$ where $t \geq 0$ and T formula β.

Definition ($\approx_T$)

Let $\langle\alpha_1\rangle^i \approx_T \langle\alpha_2\rangle^j$ if $\langle\alpha_1\rangle^i \leq_T \langle\alpha_2\rangle^j$ and $\langle\alpha_2\rangle^j \leq_T \langle\alpha_1\rangle^i$.

Obviously $\approx_T$ is a equivalence relation on T formula structures.

Definitions

We define

$$\{\langle\alpha\rangle^i\}^{\approxeq}_T = \{\langle\beta\rangle^j | \langle\beta\rangle^j \approx_T \langle\alpha\rangle^i\} (i, j \geq 0)$$

Obviously

$$\{\alpha\}^{\approxeq}_T = \{\langle\beta\rangle^j | \langle\beta\rangle^j \approx_T \alpha\} (j \geq 0)$$

- $T/_{\approx_T}$ denote the set of all $\{\langle\alpha\rangle^i\}^{\approxeq}_T$ where $\langle\alpha\rangle^i \in T^s$ and $i \geq 0$.
- $T^\bullet/_{\approx_T}$ denote the set of all $\{\alpha\}^{\approxeq}_T$ where $\langle\alpha\rangle^i \in T^s$ and $i \geq 0$.

Define $\{\langle\alpha\rangle^i\}^{\approxeq}_T \preceq_T \{\langle\beta\rangle^j\}^{\approxeq}_T$ if $\langle\alpha\rangle^i \leq_T \langle\beta\rangle^j$.

Closure operation

We define a closure operation C on $T/_{\approx_T}$ as follows:

$$C(\{\langle\alpha\rangle^i\}^{\approx}_T) = \{\underbrace{\bigwedge}_{1\leq j\leq n} \beta_j\}^{\approx}_T \quad \text{for any } \{\beta_j\}^{\approx}_T \in T^{\bullet}/_{\approx_T} \text{ s.t. } \{\langle\alpha\rangle^i\}^{\approx}_T \preceq_T \{\beta_j\}^{\approx}_T$$

Lemma

For any $\{\langle\alpha\rangle^i\}^{\approx}_T, \{\langle\beta\rangle^j\}^{\approx}_T \in T/_{\approx_T}$, the following hold:

(1) $\{\langle\alpha\rangle^i\}^{\approx}_T \preceq_T C(\{\langle\alpha\rangle^i\}^{\approx}_T)$.

(2) *if* $\{\langle\alpha\rangle^i\}^{\approx}_T \preceq_T \{\langle\beta\rangle^j\}^{\approx}_T$, *then* $C(\{\langle\alpha\rangle^i\}^{\approx}_T) \preceq_T C(\{\langle\beta\rangle^j\}^{\approx}_T)$.

(3) $C(C(\{\langle\alpha\rangle^i\}^{\approx}_T)) \preceq_T C(\{\langle\alpha\rangle^i\}^{\approx}_T)$

Interitor operation

We defined a interitor operation I on $T/_{\approx_T}$ as follows:

$$I(\{\langle\alpha\rangle^i\}^{\approx}_T) = \{ \underbrace{\bigvee}_{1\leq j\leq n} \beta_j\}^{\approx}_T \quad \text{for any } \{\beta_j\}^{\approx}_T \in T^{\bullet}/_{\approx_T} \text{ s.t. } \{\beta_j\}^{\approx}_T \preceq_T \{\langle\alpha\rangle^i\}^{\approx}_T.$$

Lemma

For any $\{\langle\alpha\rangle^i\}^{\approx}_T, \{\langle\beta\rangle^j\}^{\approx}_T \in T/_{\approx_T}$, the following hold:

(1) $I(\{\langle\alpha\rangle^i\}^{\approx}_T) \preceq_T \{\langle\alpha\rangle^i\}^{\approx}_T$.

(2) *if* $\{\langle\alpha\rangle^i\}^{\approx}_T \preceq_T \{\langle\beta\rangle^j\}^{\approx}_T$, *then* $I(\{\langle\alpha\rangle^i\}^{\approx}_T) \preceq_T I(\{\langle\beta\rangle^j\}^{\approx}_T)$.

(3) $I(\{\langle\alpha\rangle^i\}^{\approx}_T) \preceq_T I(I(\{\langle\alpha\rangle^i\}^{\approx}_T))$

Modal operations

We define a unary operations $\Diamond$ and $\Box$ on $T/_{\approx_T}$:

$$\Diamond\{\langle\alpha\rangle^i\}^{\approx}_T = \{\langle\alpha\rangle^{i+1}\}^{\approx}_T$$

$$\Box\{\langle\alpha\rangle^i\}^{\approx}_T = \{\langle\varphi\rangle^j\}^{\approx}_T \quad \text{s.t} \quad \langle\varphi\rangle^{j+1} \approx_T \langle\alpha\rangle^i$$

We define two unary operation on $T/_{\approx_T}$ as follows:

$$\blacklozenge(\{\varphi\}^{\approx}_T) = C(\Diamond(\{\varphi\}^{\approx}_T))$$

$$\blacksquare(\{\varphi\}^{\approx}_T) = I(\Box(\{\varphi\}^{\approx}_T))$$

Main Lemmas

Lemma

For any $\{\langle\alpha\rangle^i\}^{\approxeq}_T, \{\langle\beta\rangle^j\}^{\approxeq}_T \in T/_{\approx_T}$, $\Diamond C(\{\langle\alpha\rangle^i\}^{\approxeq}_T) \preceq_T C(\Diamond\{\langle\alpha\rangle^i\}^{\approxeq}_T)$.

Lemma

For any $\{\varphi\}^{\approxeq}_T \in T/_{\approx_T}$, *the following hold:*

(1) $\blacklozenge\blacklozenge(\{\varphi\}^{\approxeq}_T) \preceq_T \blacklozenge(\{\varphi\}^{\approxeq}_T)$

(2) $(\{\varphi\}^{\approxeq}_T) \preceq_T \blacklozenge(\{\varphi\}^{\approxeq}_T)$

(3) *If* $\blacklozenge((\{\varphi\}^{\approxeq}_T)) \preceq_T \{\psi\}^{\approxeq}_T$, *then* $\blacklozenge((\{\neg\psi\}^{\approxeq}_T)) \preceq_T \{\neg\varphi\}^{\approxeq}_T$

Lemma

For any $\{\varphi\}^{\approxeq}_T, \{\psi\}^{\approxeq}_T \in T/_{\approx_T}$, $\blacklozenge(\{\varphi\}^{\approxeq}_T) \preceq_T \{\psi\}^{\approxeq}_T$ *iff* $\{\varphi\}^{\approxeq}_T \preceq_T \blacksquare\{\psi\}^{\approxeq}_T$.

Finite algebras

For any $\{\varphi\}_T^{\approxeq}, \{\psi\}_T^{\approxeq} \in T/_{\approx_T}$ one defines:

- $\{\varphi\}_T^{\approxeq} \wedge \{\psi\}_T^{\approxeq} = \{\varphi \wedge \psi\}_T^{\approxeq}$,
- $\{\varphi\}_T^{\approxeq} \vee \{\psi\}_T^{\approxeq} = \{\varphi \vee \psi\}_T^{\approxeq}$
- $\neg\{\varphi\}_T^{\approxeq} = \{\neg\varphi\}_T^{\approxeq}$

Lemma

$\mathbf{A}(T, \Phi) = (T^\bullet/_{\approx_T}, \wedge, \vee, \neg, \blacklozenge, \blacksquare)$ *is a finite tqBa5.*

SFMP for tqBa5 I

We define a assignment σ from T-formulae to $\mathbf{A}(T, \Phi)$ as follows:
$\sigma(p) = \{p\}^{\approx}_{T}$ for any $p \in T$.

Lemma

$\sigma(\varphi) = \{\varphi\}^{\approx}_{T}$

Lemma

$\Diamond^{i}\{\varphi\}^{\approx}_{T} \preceq_{T} \Diamond^{i}_{c}\{\varphi\}^{\approx}_{T}$

Lemma

If $\Phi \nvdash_{G5} \langle\varphi\rangle^{i} \Rightarrow_{T} \psi$, *then* $\Phi \not\models_{\mathbf{A(T,\blacksquare)}} \sigma(\langle\varphi\rangle^{i}) \preceq_{T} \sigma(\psi)$

SFMP fot tqBa 5 II

Theorem

If $\Phi \not\vdash_{G5} \langle\varphi\rangle^i \Rightarrow \psi$ then there exists a model $(\mathbf{G}, \sigma)$ s.t. $\mathbf{G}$ is finite tqBa5 such that all sequents in Φ is true while $\langle\varphi\rangle^i \Rightarrow \psi$ is not.

Theorem

The variety tqBa5 has SFMP

FEP

Theorem

The variety tqBa5 has FEP


THANK YOU