Modal Quantifiers, Potential Infinity, and Yablo sequences

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Yablo’s paradox

Arithmetization of Yablo sentences

Potentially infinite domains and sl-semantics

Modal interpretation of quantifiers

Yablo sequences with modal quantifiers

Summing up
Yablo’s paradox

\[ Y_0 \quad \text{For any } k > 0, \ Y_k \text{ is false.} \]
\[ Y_1 \quad \text{For any } k > 1, \ Y_k \text{ is false.} \]
\[ Y_2 \quad \text{For any } k > 2, \ Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \quad \text{For any } k > n, \ Y_k \text{ is false.} \]
\[ \vdots \]
Yablo’s paradox

\[ Y_0 \text{ For any } k > 0, Y_k \text{ is false.} \]
\[ Y_1 \text{ For any } k > 1, Y_k \text{ is false.} \]
\[ Y_2 \text{ For any } k > 2, Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \text{ For any } k > n, Y_k \text{ is false.} \]
\[ \vdots \]

- Suppose \( Y_n \).
- So for any \( j > n \), \( \neg Y_j \).
- So \( \neg Y_{n+1} \) and for any \( j > n + 1 \), \( \neg Y_j \).
- So \( Y_{n+1} \). Contradiction.
- So \( \neg Y_n \) unconditionally.
- So \( \exists k > n \ Y_k \). Rinse and repeat.
Background assumptions (Ketland, 2005)

Uniform disquotation
\[ \forall x \ (Y(x) \equiv Tr(Y(x))) \]

Local disquotation
For any particular \( n \), assume \( Y(\bar{n}) \equiv Tr(Y(\bar{n})) \).

\( \omega \)-rule
If for any \( n \ \varphi(\bar{n}) \), derive \( \forall x \ \varphi(x) \).
Finitistic way out?

The idea
If the world is finite, there are only finitely many Yablo sentences, and the last one is vacuously true.

The challenge
Make sense of arithmetic in a formal finitistic setting.

The strategy
There could be more things: potential infinity.
Existence of Yablo Formulas (Priest, 1997)

Definition (Yablo formula)

$Y(x)$ is a Yablo formula in $T$ iff
\[ T \vdash \forall x (Y(x) \equiv \forall w > x \neg \text{Tr}(\text{⌜Y(\check{w})⌝})) \]

Yablo sentences are of the form $Y(\check{n})$.

Theorem

If $T$ is nice, there exists a Yablo formula in $T$. 
Existence of Yablo Formulas (Priest, 1997)

Definition (Yablo formula)

\( Y(x) \) is a Yablo formula in \( T \) iff

\[ T \vdash \forall x (Y(x) \equiv \forall w > x \neg \text{Tr}(\overline{\neg Y(\bar{w})})]. \]

Yablo sentences are of the form \( Y(\bar{n}) \).

Theorem

*If \( T \) is nice, there exists a Yablo formula in \( T \).*

Proof.

- Let \( \varphi(x, y) = \forall w > x \neg \text{Tr}(\text{sub}(y, \overline{\neg y}, \text{name}(w))) \).
- By the Diagonal Lemma, there is a formula \( Y(x) \) s.t.:
  \[ T \vdash Y(x) \equiv \forall w > x \neg \text{Tr}(\text{sub}(\overline{\neg Y(x)}, \overline{\neg y}, \text{name}(w))). \]
- \( T \vdash Y(x) \equiv \forall w > x \neg \text{Tr}(\overline{\neg Y(\bar{w})}). \)
\(\omega\)-inconsistency of Yablo formulas (Ketland, 2005)

Definition (\(\omega\)-consistency)

T is \(\omega\)-consistent iff there is no \(\varphi(x)\) s.t. simultaneously:

\[
\forall n \in \omega \ T \vdash \neg \varphi(n)
\]

\[
T \vdash \exists x \varphi(x)
\]

T is \(\omega\)-inconsistent o/w.
ω-inconsistency of Yablo formulas (Ketland, 2005)

Definition (ω-consistency)

T is ω-consistent iff there is no φ(x) s.t. simultaneously:

∀ n ∈ ω  T ⊢ ¬φ(\overline{n})

T ⊢ ∃xφ(x)

T is ω-inconsistent o/w.

Definition (PA_F)

Let \mathcal{L}_F be standard language extended with F.

PA_F := PA ∪ \{F(\overline{n}) \equiv ∀ x > \overline{n} ¬F(x) : n ∈ ω\}
\(\omega\)-inconsistency of Yablo formulas (Ketland, 2005)

**Definition (\(\omega\)-consistency)**

T is \(\omega\)-consistent iff there is no \(\varphi(x)\) s.t. simultaneously:
\[
\forall n \in \omega \; T \vdash \neg \varphi(n) \quad \text{TVD} \\
T \vdash \exists x \varphi(x)
\]

T is \(\omega\)-inconsistent o/w.

**Definition (\(\text{PA}_F\))**

Let \(\mathcal{L}_F\) be standard language extended with \(F\).

\[\text{PA}_F := \text{PA} \cup \{F(n) \equiv \forall x > n \neg F(x) : n \in \omega\}\]

**Theorem**

\(\text{PA}_F\) is \(\omega\)-inconsistent.
$\omega$-inconsistency of Yablo formulas (Ketland, 2005)

\[ \text{PA}_F := \text{PA} \cup \{F(n) \equiv \forall x > n \neg F(x) : n \in \omega\} \]

- Work in $\text{PA}_F$. Fix an $n \in \omega$ and assume $F(n)$.  
  \[ \forall x > n \neg F(x). \tag{\star} \]
ω-inconsistency of Yablo formulas (Ketland, 2005)

\[ \text{PA}_F := \text{PA} \cup \{ F(\bar{n}) \equiv \forall x > \bar{n} \neg F(x) : n \in \omega \} \]

- Work in \( \text{PA}_F \). Fix an \( n \in \omega \) and assume \( F(\bar{n}) \).

\[ \forall x > \bar{n} \neg F(x). \]  \hfill (★)

- In particular, \( \forall x > n + 1 \neg F(x) \).
- This is equivalent to \( F(n + 1) \).
- But from (★), \( \neg F(\bar{n} + 1) \) follows. Contradiction.
- So unconditionally \( \neg F(\bar{n}) \):

\[ \forall n \in \omega \quad \text{PA}_F \vdash \neg F(\bar{n}). \]  \hfill (1)

- By definition of \( \text{PA}_F \):

\[ \forall n \in \omega \quad \text{PA}_F \vdash \exists x > \bar{n} F(x). \]

- In particular:

\[ \text{PA}_F \vdash \exists x F(x). \]  \hfill (2)
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^\mathcal{M} = A$.
- $\forall n \in \omega (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x F(x)$.
- Moreover, $\forall n \in \omega (\mathcal{M}, A) \models \exists x > n F(x)$.
- Hence $\forall n \in \omega (\mathcal{M}, A) \models F(n) \equiv \forall x > n \neg F(x)$ (both sides are false).
- So $(\mathcal{M}, A) \models \text{PA}_F$ and $\text{PA}_F$ is consistent.
Adding local disquotation

Definition

\[ AD = \{ \text{Tr}(\overline{\phi}) \equiv \phi : \phi \in \text{Sent}_L \} \]

\[ YD = \{ \text{Tr}(\overline{\text{Y}(\overline{n})}) \equiv \text{Y}(\overline{n}) : \text{Y}(\overline{n}) \text{ belongs to the Yablo sequence} \} \]
Adding local disquotation

Definition

\[ AD = \{ \text{Tr}(\overline{\varphi}) \equiv \varphi : \varphi \in \text{Sent}_L \} \]

\[ YD = \{ \text{Tr}(\overline{Y(n)}) \equiv Y(n) : Y(n) \text{ belongs to the Yablo sequence} \} \]

Definition (PA_D)

PAT is obtained from PA by adding Tr. (induction!)

\[ PA_D = PAT \cup AD \cup YD. \]

\[ PA_D^- \text{ is } PA_D \text{ with induction without } \text{Tr}. \]
Adding local disquotation

Theorem

$\text{PA}_D$ is $\omega$-inconsistent.

Proof.

- Existence of $YF$ entails:
  \[ \forall n \in \omega \; \text{PA}_D \vdash Y(n) \equiv \forall x > n \neg \text{Tr}(\Gamma Y(x)^\dagger). \]

- By the inclusion of $YD$ we get:
  \[ \forall n \in \omega \; \text{PA}_D \vdash \text{Tr}(\Gamma Y(n)^\dagger) \equiv \forall x > n \neg \text{Tr}(\Gamma Y(x)^\dagger). \]

- Let $F(x) := \text{Tr}(\Gamma Y(x)^\dagger)$:
  \[ \forall n \in \omega \; \text{PA}_D \vdash F(n) \equiv \forall x > n \neg F(x). \]

- So $\text{PA}_D$ contains $\text{PA}_F$ (which is $\omega$-inconsistent).
The consistency of $\text{PA}_D^-$

**Theorem**

$\text{PA}_D^-$ *is consistent.*

**Proof.**

- Take a nonstandard $\mathcal{M}$ of $\text{PA}$.
- Let $t(x) := \neg \exists x > n \text{ Tr}(\neg Y(x))$. By overspill, there are nonstandard $b$ and $c$ such that $t^\mathcal{M}(b) = c$.
- Let $\text{Tr}^\mathcal{M} = S = \text{Th}_\mathcal{L}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.
  
  $$\forall n \in \omega \; (\mathcal{M}, S) \models \exists x > n \text{ Tr}(\neg Y(x))$$

  $$\forall n \in \omega \; (\mathcal{M}, S) \models \neg Y(n)$$

- Standard $Y(n)$ are not in $S$, so:
  
  $$\forall n \in \omega \; (\mathcal{M}, S) \models \neg \text{Tr}(\neg Y(n)).$$

- So $(\mathcal{M}, S) \models YD$ (UYD fails here).
The consistency of $\text{PA}_D$

**Theorem**

$\text{PA}_D$ is consistent.

**Proof.**

By finite satisfiability (put only the last Yablo sentence in the extension of $Tr$, check induction holds), and compactness.
Conservativeness of $\text{PA}_D$

Theorem

$\text{PA}_D$ is a conservative extension of $\text{PA}$.
Conservativeness of $\text{PA}_D$

**Theorem**

$\text{PA}_D$ is a conservative extension of $\text{PA}$.

**Proof.**

- Suppose $\text{PA} \not\vdash \varphi$.
- So $\text{PA} \cup \{\neg \varphi\}$ is consistent.
- For a nonstandard $\mathcal{M}$ of $\text{PA}$, $\mathcal{M} \models \neg \varphi$.
- There is an elementarily equivalent $\mathcal{M}'$ such that $(\mathcal{M}', \text{Tr}^{\mathcal{M}'}) \models \text{PA}_D$.
- $(\mathcal{M}', \text{Tr}^{\mathcal{M}'}) \not\models \varphi$, and so $\text{PA}_D \not\vdash \varphi$. 

Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (Tr(\neg \neg Y(x)) \equiv Y(x)) \]
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x(\text{Tr}(\overline{\overline{\overline{\overline{Y(x)}}}}) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + \text{UYD} \). \( S \) is inconsistent.
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (\text{Tr}(\neg Y(x)) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.

Work in \( S \).

- \( \forall x (Y(x) \equiv \forall w > x \neg \text{Tr}(\neg Y(w))) \) [Yablo existence]
- UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
- So \( \forall x (Y(x) \equiv \forall w > x \exists z > w \text{Tr}(\neg Y(z))) \) [unraveling]
- By UYD: \( \forall x (Y(x) \equiv \forall w > x \exists z > w Y(z)) \)
- So \( \forall x (Y(x) \equiv \exists w > x Y(w)) \)
- \( \forall x ((\forall w > x \neg Y(w)) \equiv (\exists w > x Y(w))) \)
Local disquotation with $\omega$-rule is inconsistent

Theorem

Let $\text{PA}^\omega_D = (\text{PAT}^- \cup \text{AD} \cup \text{YD})^\omega$. $\text{PA}^\omega_D$ is inconsistent.

(AD is not needed.)
Local disquotation with $\omega$-rule is inconsistent

Theorem

Let $PA_D^{\omega-} = (PAT^- \cup AD \cup YD)^\omega$. $PA_D^{\omega-}$ is inconsistent. ($AD$ is not needed.)

Proof idea.

- $\forall n \in \omega \quad PA_D^{\omega-} \vdash \neg Y(n)$ [internalized standard reasoning]
- $\forall n \in \omega \quad PA_D^{\omega-} \vdash \neg Tr(\overline{\Gamma Y(n)})$ [$Y$ disquotation]
- $PA_D^{\omega-} \vdash \forall x \neg Tr(\overline{\Gamma Y(\dot{x})})$ [$\omega$-rule]
- In particular: $PA_D^{\omega-} \vdash Y(23)$
Classical set-up vs. Yablo

- Even those theories which prove the existence of Yablo sentences are still consistent.
- They’re $\omega$-inconsistent with local Yablo disquotation, though.
- One way to obtain a contradiction: uniform Yablo disquotation.
- Another one: local disquotation and $\omega$ – rule.
sl-semantics (Mostowski, 2001a,b, 2016)

Definition (FM-domains)

Take a relational arithmetical language.

\[ FM(\mathbb{N}) = \{ \mathbb{N}_n : n = 1, 2, ... \} \]

\[ \mathbb{N}_n = (\{0, 1, ..., n - 1\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)}) \]
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Definition (sl-theory of FM(\mathbb{N}))

- Satisfaction in finite points in FM(\mathbb{N}) is standard.
- \( FM(\mathbb{N}) \models_{sl} \varphi \) iff \( \exists m \forall k \ (k \geq m \Rightarrow \mathbb{N}_k \models \varphi) \)
- \( sl(FM(\mathbb{N})) = \{ \varphi \in \text{Sent}_L : FM(\mathbb{N}) \models_{sl} \varphi \} \)
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- sl(FM(\mathbb{N})) = \{ \varphi \in Sent_L : FM(\mathbb{N}) \models_{sl} \varphi \}

Definition (FM(\mathbb{N})^T)

An \( FM(\mathbb{N})^T \)-domain is a set of \((\mathbb{N}_k, T_k)\) containing a unique member for each \( k \in \omega \), where \( T_k \subseteq \{0, \ldots, k - 1\} \).
Things that are kinda the same

- Syntax is still representable.
- Truth is still undefinable.
- Diagonal lemma still $sl$-holds.

**Theorem (sl-Yablo existence)**

There exists a formula $Y(x)$ s.t. for any $FM(\mathbb{N})^T$-domain:

$$\forall n \in \omega \; FM(\mathbb{N})^T \models_{sl} Y(n) \equiv \forall x \,(x > n \Rightarrow \neg Tr(\neg Y(x)))$$
YS are non-trivially false in the limit

Theorem

For any class $\mathcal{K}$ of finite models, if $\mathcal{K} \models_{\text{sl}} AD + YD$, then:

$\forall n \in \omega \mathcal{K} \models_{\text{sl}} \neg Y(n)$. AD is not essential.
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**Reason.**

The standard argument still flies, *mutatis mutandis*. 
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There is an FM-domain $sl$-satisfying $AD \cup YD$. 

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Theorem

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\forall n \in \omega \ \mathcal{K} \models_{sl} \neg Y(n) \text{. AD is not essential.}
\]

Reason.

The standard argument still flies, mutatis mutandis.

Theorem

There is an FM-domain \( sl \)-satisfying \( AD \cup YD \).

Reason.

In each point take truth to refer to all existing codes of true arithmetical formula, and the code of the last YS.
There is no free lunch

Theorem

*There is an FM-domain sl-satisfying AD ∪ YD.*
There is no free lunch

Theorem
There is an FM-domain sl-satisfying AD ∪ YD.

Theorem (The cost)
The sl-theory of this model is ω-inconsistent.
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Theorem

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Theorem (The cost)

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Reason.

• Each particular YS is sl-fails.
There is no free lunch

Theorem

*There is an FM-domain sl-satisfying $AD \cup YD$.*

Theorem (The cost)

*The sl-theory of this model is $\omega$-inconsistent.*

Reason.

- Each particular YS is sl-fails.
- In each finite point, the last YS is satisfied.
There is no free lunch

Theorem

*There is an FM-domain sl-satisfying AD ∪ YD.*

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*The sl-theory of this model is ω-inconsistent.*

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- Each particular YS is sl-fails.
- In each finite point, the last YS is satisfied.
- *Some YS is true* sl-holds.
There is no free lunch

Theorem

*There is an FM-domain sl-satisfying AD ∪ YD.*

Theorem (The cost)

The *sl-theory of this model is ω-inconsistent.*

Reason.

- Each particular YS is sl-fails.
- In each finite point, the last YS is satisfied.
- *Some YS is true* sl-holds.

Fact (Cheap shot)

- *n is the greatest number* sl-fails, for any *n.*
- *The greatest number exists* sl-holds.
Modal interpretation of quantifiers (Urbaniak, 2016)

Definition (Accessibility relation in FM-domains)

\[ R(M, N) \text{ iff } M \subseteq N. \]

For \( \mathbb{N}_m, \mathbb{N}_n \in FM(\mathbb{N}) \) this boils down to \( m \leq n \).
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Definition (\( m \)-semantics)

- If \( \varphi \) is atomic, then \( (K, M) \models_m \varphi \), iff \( M \models \varphi \).
- Clauses for Boolean connectives are standard.
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- Clauses for Boolean connectives are standard.
- \( (K, M) \models_m \exists x \varphi(x) \) iff there are \( N \in K \) and \( a \in N \) s.t. \( R(M, N) \) and \( (K, N) \models_m \varphi[a] \).
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Definition (\( msl \)-theory)

\[ msl(FM(\mathbb{N})) = \{ \varphi : \exists n \forall k \ k \geq n \Rightarrow (FM(\mathbb{N}), \mathbb{N}_k) \models_m \varphi \} \]
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- Clauses for Boolean connectives are standard.
- \( (\mathcal{K}, M) \models_m \exists x \varphi(x) \) iff there are \( N \in \mathcal{K} \) and \( a \in N \) s.t. 
  \[ R(M, N) \text{ and } (\mathcal{K}, N) \models_m \varphi[a]. \]

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Example

\( (\exists x \forall y x \geq y) \in sl(FM(\mathbb{N})), \notin msl(FM(\mathbb{N})) \)
Theorem

\[ msl(FM(\mathbb{N})) = Th(\mathbb{N}) \]

Reason.

- Finite points are submodels of \( \mathbb{N} \).
- Q-free \( \varphi \) are preserved for parameters in a point.
- \( \exists x \varphi \) true in \( \mathbb{N} \) has a finite witness, which belongs to some finite point.
YS & the modal interpretation

Theorem

If $YD \subseteq msl(FM(\mathbb{N})^Y)$, then:

$$\forall n \in \omega \ Y(n) \notin msl(FM(\mathbb{N})^Y).$$
YS & the modal interpretation

Theorem

If \( YD \subseteq \text{msl}(FM(\mathbb{N})^Y) \), then:

\[
\forall n \in \omega \ Y(n) \notin \text{msl}(FM(\mathbb{N})^Y).
\]

Proof.

- Suppose \( \exists n \ Y(n) \in \text{msl}(FM(\mathbb{N})^Y) \)
- \( \exists l \ \forall k \geq l \ \mathbb{N}_k \models m \ Y(n) \)
- Pick a witness. \( \forall k \geq l \ \mathbb{N}_k \models m \ \forall x (x > n \rightarrow \neg Tr(Y(x))). \)
- \( \forall k \geq l \ \forall p \geq k \ \forall a < p \ \mathbb{N}_p \models m \ a > n \rightarrow \neg Tr(Y(a)) \)
- \( \forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models \neg Tr(Y(a)) \)
- \( YD: \forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models m \ \neg Y(a) \)
- \( \mathbb{N}_p \models m \ \exists x > a \ Tr(Y(x)) \) (content of \( \neg Y(a) \))
- \( \exists q \geq p \exists b < q \mathbb{N}_q \models m \ b > a \land Tr(Y(b)) \)
- \( \exists q \geq p \exists b \in (a, q) \mathbb{N}_q \models m \ Tr(Y(b)) \) Contradiction!
No LYD

Theorem

There is no $FM(\mathbb{N})^Y$-domain such that $YD \subseteq msl(FM(\mathbb{N})^Y)$.
Theorem

There is no $\text{FM}(\mathbb{N})^Y$-domain such that $YD \subseteq msl(\text{FM}(\mathbb{N})^Y)$.

Proof.

- Suppose o/w
- Previous theorem: $\forall n \forall l \exists k \geq l \mathbb{N}_k \not\models_m Y(n)$
- $\forall n \forall l \exists p \geq l \exists a > n \mathbb{N}_p \models_m \text{Tr}(Y(a))$ (content of YS)
- $YD: \forall n \forall l \exists p \geq l \exists a > n \mathbb{N}_p \models_m Y(a)$
- Let $n = l = 0$: $\exists p, a > 0 \forall q \geq p \mathbb{N}_q \models_m \forall x > a \neg \text{Tr}(Y(x))$
- Pick witness $a > 0$. $Y(a) \in msl(\text{FM}(\mathbb{N})^Y)$. Contradiction!
Summing up

LAD and LYD are consistent, yet $\omega$-inconsistent. Adding $\omega$-rule or UYD gives inconsistency. $sl$-semantics YS are all false, the $sl$-theory is consistent, but $\omega$-inconsistent. Also, $sl(FM(N))$ itself is $\omega$-inconsistent. $m$-semantics Arithmetic regained, adding LAD and LYD gives inconsistency. UYD or $\omega$-rule are not needed.
Summing up

Standard setting
LAD and LYD are consistent, yet $\omega$-inconsistent. Adding $\omega$-rule or UYD gives inconsistency.
Summing up

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LAD and LYD are consistent, yet $\omega$-inconsistent. Adding $\omega$-rule or UYD gives inconsistency.

**sl-semantics**
YS are all false, the sl-theory is consistent, but $\omega$-inconsistent. Also, $sl(FM(\mathbb{N}))$ itself is $\omega$-inconsistent.
Summing up

**Standard setting**
LAD and LYD are consistent, yet $\omega$-inconsistent.
Adding $\omega$-rule or UYD gives inconsistency.

**sl-semantics**
YS are all false, the $sl$-theory is consistent, but $\omega$-inconsistent.
Also, $sl(FM(\mathbb{N}))$ itself is $\omega$-inconsistent.

**m-semantics**
Arithmetic regained, adding LAD and LYD gives inconsistency.
UYD or $\omega$-rule are not needed.
Finite model *in concreto*

A finite sequence of finite books each saying that all the ones behind it are false. The last one is right.
(Or so we like to think.)


