Unification in modal logic

Philippe Balbiani

CNRS — Toulouse University
Institut de recherche en informatique de Toulouse

ICLA 2019
Indian Institute of Technology — Delhi

Institut de Recherche en Informatique de Toulouse
CNRS - INP - UT3 - UT1 - UT2J

Introduction

Unification problem in a logical system L

- Given a formula $\psi(x_1, \dots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(\varphi_1, \dots, \varphi_n)$ is in L

Admissibility problem in a logical system L

- Given a rule of inference $\frac{\varphi_1(x_1,\dots,x_n),\ \dots,\ \varphi_m(x_1,\dots,x_n)}{\psi(x_1,\dots,x_n)}$
- Determine whether for all formulas χ_1, ..., χ_n, if $\varphi_1(\chi_1, \dots, \chi_n)$, ..., $\varphi_m(\chi_1, \dots, \chi_n)$ are in L then $\psi(\chi_1, \dots, \chi_n)$ is in L

Introduction

Why solving unification problem?

Description logic: Given concept definitions $C(x_1, \ldots, x_n)$ and $D(x_1, \ldots, x_n)$

- Determine whether there are some redundancies between $C(x_1, \ldots, x_n)$ and $D(x_1, \ldots, x_n)$
- Solve $C(x_1, \ldots, x_n) \equiv D(x_1, \ldots, x_n)$

Example of a unification problem

- $C = \forall R.\forall R.A \sqcap \forall R.X$
- $D = Y \sqcap \forall R.Y \sqcap \forall R.\forall S.A$

Solution

- Replace X by A$\sqcap\forall S.A$
- Replace Y by $\forall R.A$

Introduction

Why solving unification problem?

Epistemic planning: Given variable-free epistemic formulas $\varphi(p_1, \dots, p_m)$ and $\psi(p_1, \dots, p_m)$

- Determine whether there exists a public announcement χ such that $\models \varphi \rightarrow \langle \chi ! \rangle \psi$
- Solve $\models \varphi \rightarrow \langle x ! \rangle \psi$

Example of a unification problem

- $K_1 p \wedge K_2(p \rightarrow q) \rightarrow \langle K_1 x ! \rangle K_2 q$

Solution

- Replace x by p

Introduction

If L is consistent then the following are equivalent:

- Formula $\varphi(x_1, \dots, x_n)$ is unifiable
- Rule $\frac{\varphi(x_1,\dots,x_n)}{\bot}$ is non-admissible

If L is finitary then the following are equivalent:

- Rule $\frac{\varphi_1(x_1,\dots,x_n),\dots,\varphi_m(x_1,\dots,x_n)}{\psi(x_1,\dots,x_n)}$ is admissible
- Formulas $\psi(\chi_1, \dots, \chi_n)$ is in L for each maximal unifiers $(\chi_1, \dots, \chi_n)$ of formulas $\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n)$

Introduction

Unification: two examples

- The formula $\Box\neg x \vee \Box x$ is unifiable in modal logic K
- The formula $x \rightarrow \Box x$ is unifiable in modal logic K

In Classical Logic

- Unification **is equivalent to** satisfiability
- Why ? Use the inference rule of Uniform Substitution

In Modal Logic

- Unification in $S4$, $S5$, etc **is not equivalent to** satisfiability
- Why ? Consider the formula $\Diamond x \wedge \Diamond\neg x$ and use the inference rule of Uniform Substitution

Introduction

In Intuitionistic Propositional Logic

The following rules are admissible but not derivable

- $\dfrac{\neg x \to y \vee z}{(\neg x \to y) \vee (\neg x \to z)}$ — **Harrop rule (1960)**
- $\dfrac{(\neg\neg x \to x) \to (x \vee \neg x)}{\neg\neg x \vee \neg x}$ — **Lemmon-Scott rule**
- $\dfrac{(x \to y) \to (x \vee \neg y)}{\neg\neg x \vee \neg y}$ — **generalized Lemmon-Scott rule**
- $\dfrac{(x \to y) \to x \vee z}{((x \to y) \to x) \vee ((x \to y) \to z)}$ — **Mints rule (1972)**

In $S4$

The following rule is is admissible but not derivable

- $\dfrac{\Box(\Box(\Box\Diamond\Box x \to x) \to (\Box x \vee \Box\neg\Box x))}{\Box\Diamond\Box x \vee \Box\neg\Box x}$

Introduction

About Classical Propositional Logic

Classical Propositional Logic is **structurally complete**

- Thus, admissibility in Classical Propositional Logic is decidable

About intermediate logics

Rybakov (1981): If L is an intermediate logic then the following are equivalent

- Rule $\mathcal{R}$ is admissible in L
- The modal translation of rule $\mathcal{R}$ is admissible in the greatest modal companion of L

Introduction

Rybakov (1982)

- The admissibility problem in extensions of $S4.3$ is decidable

Rybakov (1984)

- The admissibility problem in $S4$ is decidable

Chagrov (1992)

- There exists a decidable normal modal logic with an undecidable admissibility problem

Wolter and Zakharyaschev (2008)

- The unification problem for any normal modal logic between K_U and $K4_U$ is undecidable

Introduction

Contents

Definitions

Substitutions

- σ: variable $x \mapsto$ formula $\sigma(x)$

Applying substitutions to formulas

- $\sigma(\varphi(x_1, \ldots, x_n)) = \varphi(\sigma(x_1), \ldots, \sigma(x_n))$

Composition of substitutions

- $\sigma \circ \tau$: variable $x \mapsto$ formula $\tau(\sigma(x))$

Definitions

Let L be a propositional logic

Equivalence relation between substitutions

- $\sigma \simeq_L \tau$ iff for all variables x, $\sigma(x) \leftrightarrow \tau(x) \in L$
- "σ and τ are L-equivalent"
- Example in Classical Propositional Logic :
  - $\sigma(x) \; = \; x \leftrightarrow y$
  - $\tau(x) \; = \; (x \wedge y) \vee (\neg x \wedge \neg y)$

Partial order between substitutions

- $\sigma \preceq_L \tau$ iff there exists a substitution μ such that $\sigma \circ \mu \simeq_L \tau$
- "σ is less specific, more general than τ in L"
- Example in Classical Propositional Logic :
  - $\sigma(x) \; = \; x \vee y$
  - $\tau(x) \; = \; (x \wedge y) \vee (\neg x \wedge \neg y)$

Definitions

Unifiers

- A substitution σ is a unifier of a formula φ iff $\sigma(\varphi) \in L$

Complete sets of unifiers

- A set Σ of unifiers of a formula φ is complete iff for all unifiers τ of φ, there exists a unifier σ of φ in Σ such that $\sigma \preceq_L \tau$

Important questions

- Given a formula, has it a unifier?
- If so, has it a minimal complete set of unifiers?
- If so, how large is this set? Is this set effectively calculable?

- Definitions
- **Boolean unification**
- Modal unification
- Unification types in modal logics
- Unification in description logics
- Recent advances

Boolean unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$

Abbreviations for $\top$, $\wedge$, etc

- As usual

Example of a Boolean unification problem

- $(x \leftrightarrow y) \leftrightarrow (x \vee y)$

Solution

- $\sigma(x) = \top$ and $\sigma(y) = \top$

Boolean unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$

Abbreviations for $\top$, $\wedge$, etc

- As usual

Example of a Boolean unification problem

- $(x \to y) \wedge (\neg x \to z)$

Solutions

- $\sigma(x) = \bot$, $\sigma(y) = y$ and $\sigma(z) = \top$
- $\sigma(x) = \top$, $\sigma(y) = \top$ and $\sigma(z) = z$
- $\sigma(x) = x \wedge y$, $\sigma(y) = (x \wedge y) \vee (y \wedge z)$ and $\sigma(z) = x \wedge y \to z$

Boolean unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$

Abbreviations for $\top$, $\wedge$, etc

- As usual

Example of a Boolean unification problem

- $(x \to p) \wedge (q \to y)$

Solutions

- $\sigma(x) = \bot$ and $\sigma(y) = \top$
- $\sigma(x) = p$ and $\sigma(y) = q$
- $\sigma(x) = p \wedge x$ and $\sigma(y) = q \vee y$

Boolean unification

Proposition

Without parameters, Boolean unification **is *NP*-complete**

- $\varphi(\bar{x})$ is *CPL*-unifiable $\iff \exists\bar{x}\varphi(\bar{x})$ is *QBF*-valid

With parameters, Boolean unification **is Π_2^P-complete**

- $\varphi(\bar{p},\bar{x})$ is *CPL*-unifiable $\iff \forall\bar{p}\exists\bar{x}\varphi(\bar{p},\bar{x})$ is *QBF*-valid

Baader (1998)

Boolean unification

Projective formulas

- A formula φ is said to be **projective** iff it has a unifier σ such that $\varphi \to (\sigma(x) \leftrightarrow x)$ is in *CPL*

Any unifier σ of φ satisfying the above condition is called a **projective unifier** of φ

Lemma Projective unifiers **are closed under compositions**

Lemma Projective unifiers **are most general unifiers**

Boolean unification

Lemma Unifiable formulas **are projective**

Proof: Consider a unifier σ of φ

- Let ϵ be the substitution such that $\epsilon(x) = (\varphi \wedge x) \vee (\neg\varphi \wedge \sigma(x))$
- **Fact:**
  1. $\varphi \to (\epsilon(\psi) \leftrightarrow \psi)$ is in *CPL*
  2. $\neg\varphi \to (\epsilon(\psi) \leftrightarrow \sigma(\psi))$ is in *CPL*
- Thus, ϵ is a projective unifier of φ

Boolean unification

Lemma Projective unifiers **are closed under compositions**

Lemma Projective unifiers **are most general unifiers**

Lemma Unifiable formulas **are projective**

Proposition Boolean unification **is unitary,** i.e. every unifiable formula has a most general unifier

Boolean unification

- **Baader, F.:** *On the complexity of Boolean unification.* Information Processing Letters **67** (1998) 215–220.
- **Baader, F., Ghilardi, S.:** *Unification in modal and description logics.* Logic Journal of the IGPL **19** (2011) 705–730.
- **Martin, U., Nipkow, T.:** *Boolean unification — the story so far.* Journal of Symbolic Computation **7** (1989) 275–293.

- Definitions
- Boolean unification
- **Modal unification**
- Unification types in modal logics
- Unification in description logics
- Recent advances

Modal unification

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Abbreviation

- $\Diamond\varphi ::= \neg\Box\neg\varphi$

Examples of modal unification problems

- $\Box\neg x \vee \Box x$
- $x \to \Box x$
- $(x \to p) \wedge (x \to \Box(p \to x))$

Modal unification

Semantics

- **Frame:** directed graph $\mathcal{F} = (W, R)$
- **Models:** $\mathcal{M} = (W, R, V)$ where $V\colon x \mapsto V(x) \subseteq W$

Truth conditions in a model

- $\mathcal{M}, s \models x$ iff $s \in V(x)$
- $\mathcal{M}, s \not\models \bot$
- $\mathcal{M}, s \models \neg\varphi$ iff $\mathcal{M}, s \not\models \varphi$
- $\mathcal{M}, s \models \varphi \vee \psi$ iff $\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$
- $\mathcal{M}, s \models \Box\varphi$ iff $\forall t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Modal unification

Semantics

- **Frame:** directed graph $\mathcal{F} = (W, R)$
- **Models:** $\mathcal{M} = (W, R, V)$ where $V \colon x \mapsto V(x) \subseteq W$

Validity in a frame

- φ is **valid** in frame $\mathcal{F}$ iff φ is true at every node of every model based on $\mathcal{F}$

Modal unification

Normal modal logics

- A set *L* of formula is a normal modal logic iff
  1. *L* contains all tautologies
  2. *L* contains $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  3. *L* is closed under modus ponens: $\frac{\varphi,\ \varphi\to\psi}{\psi}$
  4. *L* is closed under uniform substitution: $\frac{\varphi}{\sigma(\varphi)}$
  5. *L* is closed under generalization: $\frac{\varphi}{\Box\varphi}$

Modal unification

Examples of normal modal logics

- Least normal modal logic: K
- Additional axiom D: $\Diamond\top$
- Additional axiom T: $\Box\varphi \to \varphi$
- Additional axiom 4: $\Box\varphi \to \Box\Box\varphi$
- Additional axiom 5: $\Diamond\varphi \to \Box\Diamond\varphi$

Modal unification

Remark that the following statements are equivalent

- Formula $\varphi(x_1, \ldots, x_n)$ is unifiable
- Rule $\frac{\varphi(x_1,\ldots,x_n)}{\bot}$ is non-admissible

Thus, unification can be reduced to non-admissibility

Ghilardi (1999) observed that in many normal modal logics L,
Admissibility can be reduced to unification

- Assume that for a unifiable formula φ, **one can compute a finite complete set Σ of unifiers**
- Thus, to decide whether the rule $\frac{\varphi}{\psi}$ is admissible in L, **it is enough to enumerate Σ and to check whether $\sigma(\psi)$ is in L for all σ in Σ**

Modal unification

Lemma The unification problem **is trivially decidable (*NP*-complete)** for any normal modal logic containing $\Diamond\top$

- KD, KT, $S4$, $S4.3$, $S5$

From the results of **Rybakov 1984, 1997**

- The unification and admissibility problems **are decidable** for intuitionistic logic, GL and $S4$

From the results of **Jeřábek 2005, 2007**

- The admissibility problem **is *coNEXPTIME*-complete** for intuitionistic logic, GL and $S4$

Modal unification

From the results of **Chagrov 1992**

- Only one — rather artificial — example of a **decidable** unimodal logic for which the admissibility problem is **undecidable**

Modal unification

Admissibility in $Alt_1 \times Alt_1$ is undecidable

Syntax

- $\varphi ::= x \mid \bot\neg\varphi \mid (\varphi \vee \psi) \mid [h]\varphi \mid [v]\varphi$

Abbreviations

- $\langle h\rangle\varphi ::= \neg[h]\neg\varphi$
- $\langle v\rangle\varphi ::= \neg[v]\neg\varphi$

Modal unification

Admissibility in $Alt_1 \times Alt_1$ is undecidable

Semantics

- **Frame:** grid $\mathcal{F} = (I, J)$ where $I, J \geq 1$
- **Models:** $\mathcal{M} = (I, J, V)$ where V:
  $x \mapsto V(x) \subseteq \{1, \ldots, I\} \times \{1, \ldots, J\}$

Truth conditions in a model

- $\mathcal{M}, (i, j) \models x$ iff $(i, j) \in V(x)$
- $\mathcal{M}, (i, j) \models [h]\varphi$ iff if $i < I$ then $\mathcal{M}, (i + 1, j) \models \varphi$
- $\mathcal{M}, (i, j) \models [v]\varphi$ iff if $j < J$ then $\mathcal{M}, (i, j + 1) \models \varphi$

Satisfiability is

- ***NP*-complete**

Modal unification

Admissibility in $Alt_1 \times Alt_1$ is undecidable

Semantics

- **Frame:** grid $\mathcal{F} = (I, J)$ where $I, J \geq 1$
- **Models:** $\mathcal{M} = (I, J, V)$ where V:
  $x \mapsto V(x) \subseteq \{1, \ldots, I\} \times \{1, \ldots, J\}$

Truth conditions in a model

- $\mathcal{M}, (i, j) \models x$ iff $(i, j) \in V(x)$
- $\mathcal{M}, (i, j) \models [h]\varphi$ iff if $i < I$ then $\mathcal{M}, (i + 1, j) \models \varphi$
- $\mathcal{M}, (i, j) \models [v]\varphi$ iff if $j < J$ then $\mathcal{M}, (i, j + 1) \models \varphi$

Admissibility is

- **undecidable**

Modal unification

Admissibility in $Alt_1 \times Alt_1$ is undecidable

Computability of admissibility

- **undecidable**

Proof

- Reduction of the domino-tiling problem $(\Delta, H, V, \Delta_u, \Delta_d, \Delta_l, \Delta_r)$ where
  - Δ is a finite set of domino-types
  - H and V are binary relations on Δ
  - $\Delta_u, \Delta_d, \Delta_l, \Delta_r$ are subsets of Δ

Modal unification

K_u: **least normal modal logic** with the **universal modality**

$K4_u$: **least normal modal logic** with the **universal modality** that contains the **extra formula**

- $\Box x \rightarrow \Box\Box x$

From the results of **Wolter and Zakharyaschev 2008**

- The unification problem for modal logics between K_u and $K4_u$ **is undecidable**

Modal unification

The unification and admissibility problems for K itself . . .

- . . . still remain **open**

Unfortunately, nothing is known about

- **The decidability status** of the unification and admissibility problems for
  - Basic modal logic K
  - Various multimodal logics
  - Various hybrid logics
  - Various description logics

- Definitions
- Boolean unification
- Modal unification
- **Unification types in modal logics**
- Unification in description logics
- Recent advances

Unification types in modal logics

Unification types in propositional logic

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type unitary (1)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- $Card(\Sigma) = 1$

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type finitary (ω)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- $Card(\Sigma) \neq 1$ but Σ is finite

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type infinitary (∞)** for L iff

- There exists a complete minimal set Σ of L-unifiers of φ
- Σ is infinite

Let L be a propositional logic and φ be a formula

An ***L*-unifier** of φ is a substitution σ such that

- $\sigma(\varphi) \in L$

We shall say that φ is of **type nullary (0)** for L iff

- There exists no complete minimal set of L-unifiers of φ

Let L be a propositional logic

We shall say that L is of **type unitary/finitary** iff

- For all formulas φ, φ is of **type unitary/finitary** for L

Examples

- Unification in **classical propositional logic** **is unitary**
- Unification in **intuitionistic propositional logic** **is finitary**

Let L be a propositional logic

We shall say that L is of **type infinitary/nullary** iff

- There exists a formula φ such that φ is of **type infinitary/nullary** for L

Example

- Unification in **modal logic** K **is nullary**

Unification types in modal logics

Unification in intuitionistic propositional logic

We have seen: *CPL*-unification **is unitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of *CPL*-unifiers **is at most** 1

Ghilardi (1999) has demonstrated that *IPL*-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of *IPL*-unifiers **is finite**

Example The formulas $x \vee \neg x$ **is *IPL*-unifiable** with the 2 following most general *IPL*-unifiers

- $\sigma(x) = \bot$
- $\tau(x) = \top$

We have seen:

- The **complexity** of Boolean unification **is *NP*-complete**

It can be easily proved that:

- The **complexity** of *IPL*-unification **is *NP*-complete too**

Lemma For all formulas φ, the following statements are equivalent:

- φ **is *IPL*-unifiable**
- φ **is *CPL*-unifiable**

Proposition *IPL*-unification **is *NP*-complete**
Proof: By the above Lemma

Remark For *IPL*-unification with constants, see

- **Rybakov, V.:** *Rules of inference with parameters for intuitionistic logic.* The Journal of Symbolic Logic **57** (1992) 912–923.

Proposition (Ghilardi 1999) *IPL*-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of *IPL*-unifiers **is finite**

Proof: We will demonstrate a similar result for $K4$

Modal logic $K4$

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^+\varphi ::= \varphi \wedge \Box\varphi$
- Semantics
  - **Frame:** directed graph $\mathcal{F} = (W, R)$ where R is transitive
  - **Model:** $\mathcal{M} = (W, R, V)$ where $V\colon p \mapsto V(p) \subseteq W$

Modal logic $K4$

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{+}\varphi ::= \varphi \wedge \Box\varphi$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, x \models p$ iff $x \in V(p)$
  - $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in W$, if xRy then $\mathcal{M}, y \models \varphi$

Proposition (Rybakov 1984, 1997) $K4$-unification **is decidable**

Proof: Later

Proposition (Ghilardi 2000) $K4$-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of $K4$-unifiers **is finite**

Proof: Later

A formula of the kind $\Box^+\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of $\Box^+\varphi$
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

Remark This definition resembles the definition of a unifier being **transparent** as used in

- **Dzik, W.:** *Transparent unifiers in modal logics with self-conjugate operators.* Bulletin of the Section of Logic **35** (2006) 73–83.

A formula of the kind $\Box^{+}\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of $\Box^{+}\varphi$
2. $\Box^{+}\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

Example To see that the formula $\Box^{+}\varphi = \Box^{+}x$ **is projective,** it suffices to consider the substitution $\sigma(x) = \top$

- $\sigma(\Box^{+}\varphi) = \Box^{+}\top$
- $\Box^{+}\varphi \to (x \leftrightarrow \sigma(x)) = \Box^{+}x \to (x \leftrightarrow \top)$

A formula of the kind $\Box^{+}\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of $\Box^{+}\varphi$
2. $\Box^{+}\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

Remark The following statements hold:

- Such σ **is a most general $K4$-unifier** for $\Box^{+}\varphi$
- $\Box^{+}\varphi \to (\psi \leftrightarrow \sigma(\psi)) \in K4$ **for each formula $\psi(x_1, \ldots, x_n)$**
- The set of all substitutions satisfying condition 2 **is closed under compositions**

A formula of the kind $\Box^+\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of $\Box^+\varphi$
2. $\Box^+\varphi \rightarrow (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \rightarrow x_i$ if $i \in A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$

Remark The substitution θ_φ^A **satisfies condition** 2

Unification types in modal logics
Unification in $K4$

A formula of the kind $\Box^+\varphi(x_1, \ldots, x_n)$ is said to be **projective** iff **there exists a substitution** σ such that

1. σ is a $K4$-unifier of $\Box^+\varphi$
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in K4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \to x_i$ if $i \in A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$

Given an arbitrary enumeration $A_1, \ldots, A_{2^n}$ of the subsets of $\{1, \ldots, n\}$, let $\theta_\varphi = \theta_\varphi^{A_1} \circ \ldots \circ \theta_\varphi^{A_{2^n}}$

Proposition For all formulas of the kind $\Box^+\varphi(x_1, \ldots, x_n)$, if $d = depth(\varphi)$ and N is the number of non-$\sim_d$-equivalent models over $x_1, \ldots, x_n$, the following statements are equivalent:

- ${\theta_\varphi}^{2N}$ is a $K4$-unifier of $\Box^+\varphi$
- $\Box^+\varphi$ is projective

- **Ghilardi, S.:** *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

Corollary It is decidable to determine whether a given formula of the kind $\Box^+\varphi$ is projective

Lemma For all formulas φ and for all substitutions σ, if σ is a $K4$-unifier of φ

- There exists a formula of the kind $\Box^+\psi$, *depth*$(\psi) \leq$ *depth*(φ), such that
  - $\Box^+\psi$ is projective
  - σ is a $K4$-unifier of $\Box^+\psi$
  - $\Box^+\psi \to \varphi \in K4$

- **Ghilardi, S.:** *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

Proposition (Ghilardi 2000) $K4$-unification **is finitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of $K4$-unifiers **is finite**

Corollary $K4$-unification **is decidable**

Proof: Given a formula φ

- Determine whether there exists a formula of the kind $\Box^+\psi$, $depth(\psi) \leq depth(\varphi)$, such that
  - $\Box^+\psi$ is projective
  - $\Box^+\psi \to \varphi \in K4$

Modal logic *S5*

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
- Semantics
  - **Frame:** partition $\mathcal{F} = (W, R)$, i.e. R is an equivalence relation
  - **Model:** $\mathcal{M} = (W, R, V)$ where $V\colon x \mapsto V(x) \subseteq W$

Modal logic $S5$

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, x \models p$ iff $x \in V(p)$
  - $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in W$, if xRy then $\mathcal{M}, y \models \varphi$

Modal logic *S*5

- Syntax
  - $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
- Important properties of modal logic *S*5
  - For all formulas φ, ψ, $\Box(\Box\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \in S5$
  - For all formulas φ, ψ, $\Box(\Diamond\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Box\psi \in S5$
  - For all variable-free formulas φ, either $\neg\varphi \in S5$, or $\varphi \in S5$

Remark The unification problem **is *NP*-complete** for $S5$

Remark In case we allow extra parameters in the formulas, $S5$-unification **becomes a more serious problem**

The formula $\varphi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ with parameters $p_1, \ldots, p_m$ and variables $x_1, \ldots, x_n$ **is $S5$-unifiable** iff there exists formulas $\chi_1, \ldots, \chi_n$ such that $\varphi(p_1, \ldots, p_m, \chi_1, \ldots, \chi_n) \in S5$

Remark If $\varphi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ **is $S5$-unifiable** then there exists an $S5$-unifier **based only on parameters $p_1, \ldots, p_m$**

Remark In case we allow extra parameters in the formulas, *S*5-unification **becomes a more serious problem**

Proposition *S*5-unification with parameters **is in** Π_2^{EXP}

Claim *S*5-unification with parameters **is *coNEXPTIME*-hard**

Proposition (Dzik 2003) $S5$-unification **is unitary,** i.e.

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a minimal complete set of $S5$-unifiers **is at most 1**

Proof: Assume $\varphi(x_1, \ldots, x_n)$ is $S5$-unifiable

- Thus, there exists a ground substitution σ such that $\sigma(\varphi) \in S5$
- Let τ be the substitution defined by
  - $\tau(x_i) = \Box\varphi \to x_i$ if $\sigma(x_i) \in S5$
  - $\tau(x_i) = \Box\varphi \wedge x_i$ if $\neg\sigma(x_i) \in S5$
- It can be proved that τ **is a most general $S5$-unifier** of φ

Remark The proofs that the **unification problems in classical propositional logic and in $S5$ are unitary** are based on the

- **Fact** Given a unifiable formula $\varphi(x_1, \ldots, x_n)$,
  - There exists a unifier σ of φ such that for all i, if $1 \leq i \leq n$, $\varphi \rightarrow (x_i \leftrightarrow \sigma(x_i)) \in L$

Remark This fact is used, for example, by Dzik (**transparent unifiers**) and Ghilardi (**projective formulas**) in

- **Dzik, W.:** *Transparent unifiers in modal logics with self-conjugate operators.* Bulletin of the Section of Logic **35** (2006) 73–83.
- **Ghilardi, S.:** *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

Remark The proofs that the **unification problems in classical propositional logic and in *S*5 are unitary** are based on the

- **Fact** Given a unifiable formula $\varphi(x_1, \ldots, x_n)$,
  - There exists a unifier σ of φ such that for all i, if $1 \leq i \leq n$, $\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in L$

Remark It is true that **if L satisfies the above fact, L-unification is unitary** but the converse is not always true

- ***S*4.2*Grz*-unification is unitary** (Ghilardi 2000)
- ***S*4.2*Grz* does not satisfy the above fact** (Dzik 2006)

Modal logic K

- Syntax
  - $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{<n}\varphi ::= \Box^{0}\varphi \wedge \ldots \wedge \Box^{n-1}\varphi$ for each $n \in \mathcal{N}$
- Semantics
  - **Frame:** directed graph $\mathcal{F} = (W, R)$
  - **Model:** $\mathcal{M} = (W, R, V)$ where $V \colon p \mapsto V(p) \subseteq W$

Modal logic K

- Syntax
  - $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{<n}\varphi ::= \Box^{0}\varphi \wedge \ldots \wedge \Box^{n-1}\varphi$ for each $n \in \mathcal{N}$
- **Truth conditions** in a model $\mathcal{M} = (W, R, V)$
  - $\mathcal{M}, x \models p$ iff $x \in V(p)$
  - $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in W$, if xRy then $\mathcal{M}, y \models \varphi$

Open question **Is K-unification decidable?**

K-unification **is not unitary** since

- $\sigma_\top(x) = \top$ and $\sigma_\bot(x) = \bot$ constitute a minimal complete set of unifiers in K of the formula $\Diamond x \to \Box x$

Our purpose: demonstrate that K-unification **is nullary,** i.e.

- There exists a formula φ such that there exists no complete minimal set of K-unifiers of φ

Method (Jeřábek, 2014) Study the K-unifiers of

- $x \to \Box x$

Unification types in modal logics
Unification in K

Method (Jeřábek, 2014) Study the K-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Lemma

- σ_n **is a K-unifier** of $x \to \Box x$ for each $n \in \mathcal{N}$
- $\sigma_\top$ **is a K-unifier** of $x \to \Box x$

Unification types in modal logics

Unification in K

Method (Jeřábek, 2014) Study the K-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Lemma For all K-unifiers σ of $x \to \Box x$ and for all $n \in \mathcal{N}$, the following statements are equivalent:

- $\sigma \leq_K \sigma_n$
- $\sigma(x) \to \Box^n \bot \in K$

Method (Jeřábek, 2014) Study the K-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Lemma For all substitutions σ, the following statements are equivalent:

- $\sigma \leq_K \sigma_\top$
- $\sigma(x) \in K$

Proposition (Jeřábek, 2014) For all formulas φ, $depth(\varphi) = n$,

- If $\varphi \to \Box\varphi \in K$ then either $\varphi \to \Box^n\bot \in K$, or $\varphi \in K$

Corollary The following substitutions form a complete set of K-unifiers for the formula $x \to \Box x$

- $\sigma_n(x) = \Box^{<n}x \wedge \Box^n\bot$ for each $n \in \mathcal{N}$
- $\sigma_\top(x) = \top$

Corollary K-unification **is nullary,** i.e.

- There exists a formula φ such that there exists no complete minimal set of K-unifiers of φ

Proof: Take $\varphi = x \to \Box x$

- Definitions
- Boolean unification
- Modal unification
- Unification types in modal logics
- **Unification in description logics**
- Recent advances

Unification in description logics

Syntax of the **basic Boolean description language** $\mathcal{FL}_0$

- $C ::= A \mid \top \mid (C \sqcap D) \mid \forall R.C$ — **concept descriptions**
- A denotes an arbitrary **atomic concept**
- R denotes an arbitrary **atomic role**

Example of $\mathcal{FL}_0$-concept description

- $Woman \sqcap \forall child.Woman$

See

- **Baader, F.:** *Terminological cycles in KL-ONE-based knowledge representation languages.* In: AAAI'90 Proceedings of the eighth National conference on Artificial intelligence. AAAI Press (1990) 621–626.

Unification in description logics

Syntax of the **basic Boolean description language** $\mathcal{FL}_0$

- $C ::= A \mid \top \mid (C \sqcap D) \mid \forall R.C$ — **concept descriptions**

An **interpretation** is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where

- $\Delta^{\mathcal{I}}$ is a non-empty set — the **domain** of $\mathcal{I}$
- $\cdot^{\mathcal{I}}$ is the **interpretation function**
  - $A \mapsto A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
  - $R \mapsto R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

Unification in description logics

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The **interpretation function** $\cdot^{\mathcal{I}}$ is inductively extended to concept descriptions

- $(A)^{\mathcal{I}} = A^{\mathcal{I}}$
- $(\top)^{\mathcal{I}} = \Delta^{\mathcal{I}}$
- $(C \sqcap D)^{\mathcal{I}} = (C)^{\mathcal{I}} \cap (D)^{\mathcal{I}}$
- $(\forall R.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \colon \forall e \in \Delta^{\mathcal{I}}, \text{ if } (d, e) \in R^{\mathcal{I}} \text{ then } e \in (C)^{\mathcal{I}}\}$

Unification in description logics

Two concept descriptions C, D are **equivalent** ($C \equiv D$) iff

- $(C)^{\mathcal{I}} = (D)^{\mathcal{I}}$ holds for all interpretations $\mathcal{I}$

The concept description D **subsumes** the concept description C ($C \sqsubseteq D$) iff

- $(C)^{\mathcal{I}} \subseteq (D)^{\mathcal{I}}$ holds for all interpretations $\mathcal{I}$

Proposition Equivalence and subsumption of $\mathcal{FL}_0$-concept descriptions **can be decided in polynomial time**

Proof:

- **Levesque, H., Brachman, R.:** *Expressiveness and tractability in knowledge representation and reasoning.* Computational Intelligence **3** (1987) 78–93.

Unification in description logics

We **partition** the set of all atomic concepts into

- A set of **concept variables** — denoted $X, Y, \ldots$
- A set of **concept constants** — denoted $A, B, \ldots$

Syntax of the **basic Boolean description language** $\mathcal{FL}_0$ with variables and constants

- $C ::= X \mid A \mid \top \mid (C \sqcap D) \mid \forall R.C$ — **concept descriptions**

Unification in description logics

We **partition** the set of all atomic concepts into

- A set of **concept variables** — denoted $X, Y, \ldots$
- A set of **concept constants** — denoted $A, B, \ldots$

Now, an **interpretation** is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where

- $\Delta^{\mathcal{I}}$ is a non-empty set — the **domain** of $\mathcal{I}$
- $\cdot^{\mathcal{I}}$ is the **interpretation function**
  - $X \mapsto X^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
  - $A \mapsto A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
  - $R \mapsto R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

Unification in description logics

A **substitution** σ is a mapping from the set of all **concept variables** into the set of all $\mathcal{FL}_0$**-concept descriptions**

This mapping is inductively extended to concept descriptions

- $\sigma(A) = A$
- $\sigma(\top) = \top$
- $\sigma(C \sqcap D) = \sigma(C) \sqcap \sigma(D)$
- $\sigma(\forall R.C) = \forall R.\sigma(C)$

Unification in description logics

The substitution σ is a **unifier** of $\mathcal{FL}_0$-concept descriptions C and D iff

- $\sigma(C) \equiv \sigma(D)$

The $\mathcal{FL}_0$-concept descriptions C and D are **unifiable** iff they have a unifier

Example The substitution σ defined by

- $\sigma(X) = A \sqcap \forall S.A$ and $\sigma(Y) = \forall R.A$

is a unifier of the $\mathcal{FL}_0$-concept descriptions

- $C = \forall R.\forall R.A \sqcap \forall R.X$
- $D = Y \sqcap \forall R.Y \sqcap \forall R.\forall S.A$

Unification in description logics

A substitution is **ground** iff

- The $\mathcal{FL}_0$-concept descriptions it substitutes for the variables **do not contain variables**

Remark For all $\mathcal{FL}_0$-concept descriptions C, D, the following statements are equivalent:

- **There exists a unifier** of C and D
- **There exists a ground unifier** of C and D

Unification in description logics

Lemma For all $\mathcal{FL}_0$-concept descriptions $C_1, \ldots, C_n, D_1, \ldots, D_n$ and for all pairwise distinct roles $R_1, \ldots, R_n$, the following statements are equivalent:

- $C_1 \equiv D_1, \ldots, C_n \equiv D_n$
- $\forall R_1.C_1 \sqcap \ldots \sqcap \forall R_n.C_n \equiv \forall R_1.D_1 \sqcap \ldots \sqcap \forall R_n.D_n$

Unification in description logics

Given finite sets $S_0, \ldots, S_n, T_0, \ldots, T_n$ of words over the alphabet of role names, we consider the **equation**

- $S_0 \cup S_1 \cdot X_1 \cup \ldots \cup S_n \cdot X_n = T_0 \cup T_1 \cdot X_1 \cup \ldots \cup T_n \cdot X_n$

where

- $\cup$ stands for set union
- $\cdot$ stands for element-wise concatenation of sets of words

Examples

- $\{R\} \cup \{RS\} \cdot X = \{RSS\} \cup \{R\} \cdot X$
- $\{RR\} \cup \{RS\} \cdot Y = \{RSR, RR\} \cup \{R\} \cdot Y$

Unification in description logics

Given two $\mathcal{FL}_0$-concept descriptions C, D, let

- $X_1, \ldots, X_n$ be the concept variables that occur in C, D
- $A_1, \ldots, A_k$ be the concept constants that occur in C, D

Abbreviating $\forall R_1 . \ldots . \forall R_m .$ by $\forall R_1 \ldots R_m .$, the $\mathcal{FL}_0$-concept descriptions C, D can be rewritten

- $C \equiv \forall S_{0,1}.A_1 \sqcap \ldots \sqcap \forall S_{0,k}.A_k \sqcap \forall S_1.X_1 \sqcap \ldots \sqcap \forall S_n.X_n$
- $D \equiv \forall T_{0,1}.A_1 \sqcap \ldots \sqcap \forall T_{0,k}.A_k \sqcap \forall T_1.X_1 \sqcap \ldots \sqcap \forall T_n.X_n$

for finite sets of words $S_{0,i}, S_j, T_{0,i}, T_j$

Unification in description logics

Theorem (Baader and Narendran 2001) Let C, D be $\mathcal{FL}_0$-concept descriptions such that

- $C \equiv \forall S_{0,1}.A_1 \sqcap \ldots \sqcap \forall S_{0,k}.A_k \sqcap \forall S_1.X_1 \sqcap \ldots \sqcap \forall S_n.X_n$
- $D \equiv \forall T_{0,1}.A_1 \sqcap \ldots \sqcap \forall T_{0,k}.A_k \sqcap \forall T_1.X_1 \sqcap \ldots \sqcap \forall T_n.X_n$

The following statements are equivalent:

- The $\mathcal{FL}_0$-**concept descriptions** C and D **are unifiable**
- For all i, if $1 \leq i \leq k$, the **linear equation** $E_{C,D}(A_i)$
  - $S_{0,i} \cup S_1 \cdot X_{1,i} \cup \ldots \cup S_n \cdot X_{n,i} = T_{0,i} \cup T_1 \cdot X_{1,i} \cup \ldots \cup T_n \cdot X_{n,i}$

  has a solution

Unification in description logics

Example Let C, D be the following $\mathcal{FL}_0$-concept descriptions

- $C = \forall R.(A_1 \sqcap \forall R.A_2) \sqcap \forall R.\forall S.X_1$
- $D = \forall R.\forall S.(\forall S.A_1 \sqcap \forall R.A_2) \sqcap \forall R.X_1 \sqcap \forall R.\forall R.A_2$

Then

- $C \equiv C' = \forall\{R\}.A_1 \sqcap \forall\{RR\}.A_2 \sqcap \forall\{RS\}.X_1$
- $D \equiv D' = \forall\{RSS\}.A_1 \sqcap \forall\{RSR, RR\}.A_2 \sqcap \forall\{R\}.X_1$

The unification of C', D' leads to the two linear equations

- $\{R\} \cup \{RS\} \cdot X_{1,1} = \{RSS\} \cup \{R\} \cdot X_{1,1}$
- $\{RR\} \cup \{RS\} \cdot X_{1,2} = \{RSR, RR\} \cup \{R\} \cdot X_{1,2}$

Unification in description logics

Theorem (Baader and Narendran 2001) Solvability of linear equations can be decided in deterministic exponential time

Corollary (Baader and Narendran 2001) Solvability of unification problems in $\mathcal{FL}_0$ can be decided in deterministic exponential time

- Definitions
- Boolean unification
- Modal unification
- Unification types in modal logics
- Unification in description logics
- **Recent advances**

Recent advances

Description logic $\mathcal{EL}$

- Unification in $\mathcal{EL}$ is *NP*-complete
- Unification in $\mathcal{EL}^{-\top}$ is *PSPACE*-complete

Baader, F., Binh, N., Borgwardt, S., Morawska, B.: *Deciding unifiability and computing local unifiers in the description logic $\mathcal{EL}$ without top constructor.* Notre Dame Journal of Formal Logic **57** (2016) 443–476.

Recent advances

$KD = K + \Diamond\top$

KD is nullary

- $x \to p$
- $x \to \Box(p \to x)$

Balbiani, P., Gencer, Ç.: *KD is nullary.* Journal of Applied Non-Classical Logics **27** (2018) 196–205.

Recent advances

$KT = K + \Box\varphi \to \varphi$

KT is nullary

- $x \to p$
- $x \to \Box(q \to y)$
- $y \to q$
- $y \to \Box(p \to x)$

Balbiani, P.: *Remarks about the unification type of several non-symmetric non-transitive modal logics.* Logic Journal of the IGPL (to appear).

Recent advances

$KB = K + \varphi \rightarrow \Box\Diamond\varphi$

KB is nullary

- $x \rightarrow (\neg p \wedge \neg q \rightarrow \Box(p \wedge \neg q \rightarrow \Box(\neg p \wedge q \rightarrow \Box(\neg p \wedge \neg q \rightarrow x))))$

Balbiani, P., Gencer, Ç.: *About the unification type of simple symmetric modal logics.* Submitted for publication.

Recent advances

$$Alt_1 \; = \; K + \Diamond\varphi \to \Box\varphi$$

- Alt_1 is nullary for unification
- The unification problem (without parameters) in Alt_1 is decidable (in *PSPACE*)

Balbiani, P., Tinchev, T.: *Unification in modal logic* Alt_1. In Beklemishev, L., Demri, S., Máté, A. (editors): Advances in Modal Logic. Volume 11. College Publications (2016) 117–134.

Recent advances

Normal extensions of $K5 = K + \Diamond\varphi \to \Box\Diamond\varphi$

- These modal logics are unitary for unification

$K + \Box^k\bot$ for $k \geq 2$

- These modal logics are finitary for unification

Balbiani, P., Rostamigiv, M., Tinchev, T.: *About the unification type of some locally tabular modal logics.* Submitted for publication.

Recent advances

Unification in Dynamic Epistemic Logics

Syntax

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid K_a\varphi \mid [\varphi]\psi$

Abbreviations

- $\hat{K}_a\varphi ::= \neg K_a \neg\varphi$
- $\langle\varphi\rangle\psi ::= \neg[\varphi]\neg\psi$

Readings

- $K_a\varphi$: “agent *a* knows that φ holds”
- $[\varphi]\psi$: “if φ holds then ψ will hold after φ is announced”
- $\hat{K}_a\varphi$: “it is compatible with *a*’s knowledge that φ holds”
- $\langle\varphi\rangle\psi$: “$\varphi$ holds and ψ will hold after φ is announced”

Recent advances

Unification in Dynamic Epistemic Logics

Example of unification problems

$$\varphi(\bar{p}) \rightarrow \langle x \rangle K_a \psi(\bar{p})$$

- $\varphi(\bar{p})$ describes an initial situation
- x is the announcement
- $K_a \psi(\bar{p})$ — with $\psi(\bar{p})$ Boolean formula — is a goal formula

Other examples of unification problems

- $\varphi \rightarrow \langle x \rangle K_a \psi$
- $\varphi \rightarrow \langle x \rangle (K_{a_1} \psi_1 \wedge \ldots \wedge K_{a_n} \psi_n)$
- $\varphi \rightarrow \langle x \rangle K_{a_1} \ldots K_{a_n} \psi$
- $\varphi \rightarrow \langle K_b x \rangle K_a \psi$
- $\varphi \rightarrow \langle K_b x \rangle (K_{a_1} K_b \psi_1 \wedge \ldots \wedge K_{a_n} K_b \psi_n \wedge K_{a_1} \hat{K}_b \chi_1 \wedge \ldots \wedge K_{a_n} \hat{K}_b \chi_n)$

Conclusion

Applications to description logics

- **Baader, F., Fernández Gil, O., Morawska, B.:** *Hybrid unification in the description logic $\mathcal{EL}$.* In Fontaine, P., Ringeissein, C., Schmidt, R. (editors): Frontiers of Combining Systems. Springer (2013) 295–310.
- **Baader, F., Morawska, B.:** *Unification in the description logic $\mathcal{EL}$.* In Treinen, R. (editor): Rewriting Techniques and Applications. Springer (2009) 350–364.
- **Baader, F., Narendran, P.:** *Unification of concept terms in description logics.* Journal of Symbolic Computation **31** (2001) 277–305.

Conclusion

Applications to epistemic logics and temporal logics

- **Babenyshev, S., Rybakov, V.:** *Unification in linear temporal logic LTL.* Annals of Pure and Applied Logic **162** (2011) 991–1000.
- **Rybakov, V.:** *Logical consecutions in discrete linear temporal logic.* The Journal of Symbolic Logic **70** (2005) 1137–1149.
- **Rybakov, V.:** *Multi-modal and temporal logics with universal formula — reduction of admissibility to validity and unification.* Journal of Logic and Computation **18** (2008) 509–519.

Conclusion

Admissibility and unification in other non-classical logics

- **Cintula, P., Metcalfe, G.:** *Structural completeness in fuzzy logics.* Notre Dame Journal of Formal Logic **50** (2009) 153–182.
- **Dzik, W.:** *Unification of some substructural logics of BL-algebras and hoops.* Reports on Mathematical Logic **43** (2008) 73–3.
- **Jerábek, E.:** *Admissible rules of Łukasiewicz logic.* Journal of Logic and Computation **20** (2010) 425–447.
- **Odintsov, S., Rybakov, V.:** *Unification and admissible rules for paraconsistent minimal Johanssons logic J and positive intuitionistic logic IPC$^+$.* Annals of Pure and Applied logic **164** (2013) 771–784.

Conclusion

Proof-theoretic approaches

- **Iemhoff, R.:** *On the admissible rules of Intuitionistic Propositional Logic.* The Journal of Symbolic Logic **66** (2001) 281–294.
- **Iemhoff, R.:** *A syntactic approach to unification in transitive reflexive modal logics.* Notre Dame Journal of Formal Logic **57** (2016) 233–247.
- **Iemhoff, R., Metcalfe, G.:** *Hypersequent systems for the admissible rules of modal and intermediate logics.* In Artemov, S., Nerode, A. (editors): Logical Foundations of Computer Science. Springer (2009) 230–245.
- **Iemhoff, R., Metcalfe, G.:** *Proof theory for admissible rules.* Annals of Pure and Applied Logic **159** (2009) 171–186.

Conclusion

Decidability/complexity and proof procedures

- **Babenyshev, S., Rybakov, V., Schmidt, R., Tishkovsky, D.:** *A tableau method for checking rule admissibility in S*4*.* Electronic Notes in Theoretical Computer Science **262** (2010) 17–32.
- **Cintula, P., Metcalfe, G.:** *Admissible rules in the implication-negation fragment of intuitionistic logic.* Annals of Pure and Applied Logic **162** (2010) 162–171.
- **Ghilardi, S.:** *A resolution/tableaux algorithm for projective approximations in IPC.* Logic Journal of the IGPL **10** (2002) 229–243.
- **Jeřábek, E.:** *Complexity of admissible rules.* Archive for Mathematical Logic **46** (2007) 73–92.

Conclusion

K**-unification**

- **Jeřábek, E.:** *Blending margins: the modal logic* K *has nullary unification type.* Journal of Logic and Computation **25** (2015) 1231–1240.
- **Wolter, F., Zakharyaschev, M.:** *Undecidability of the unification and admissibility problems for modal and description logics.* ACM Transactions on Computational Logic **9** (2008) 25:1–25:20.

Conclusion

Some open problems

- Decidability of
  - parameter-free unification in modal logic K, KB ?
  - unification with parameters in modal logics KD, KDB ?
  - unification with parameters in modal logics KT, KTB ?
  - unification with parameters in modal logics Alt_1, Alt_2 ?
  - unification in implication fragments ?
- Type of
  - KB, KD, KDB, KT, KTB for parameter-free unification ?
  - $S5 \otimes S5$ and other fusions of modal logics ?
  - $S4.2 \times S4.2$ and other products of modal logics ?
  - $K + \Box^k \bot$ and other locally tabular modal logics ?
  - unification in implication fragments ?

Thank you