Paraconsistent Set Theory

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Paraconsistent Logic

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In the context of classical logic the above two notions are equivalent.

A logic is said to be **paraconsistent** if there exists a set $\Gamma$ of formulas such that $\Gamma$ is **inconsistent** but not **explosive**.
Definition

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**Definition**

A logic is called paraconsistent if there exist formulas $\varphi$ and $\psi$ such that $\{\varphi, \neg \varphi\} \not\models \psi$. 
Some Well Known Paraconsistent Logics

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For example: Jaskowski’s paraconsistent logic, Da Costa’s paraconsistent logic systems $C_n$ where $0 < n < \omega$, Priest’s logic of paradox, also other paraconsistent logics made by C. Mortensen, R. Brady, J. Marcos, W.A. Carnielli, A. Avron etc.
Syntactic developments of some paraconsistent set theories are already made by the paraconsistentists like N. C. A. da Costa, Zach Weber, Walter Carnielli, Marcelo E. Coniglio etc. Whereas the semantical developments of this area are not strongly established other than some research work done by Thierry Libert, Olivier Esser etc.


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Walter Carnielli and Marcelo E. Coniglio have talked about paraconsistent set theories $ZF_{mbC}$ and $ZFC_{il}$ in this paper. “We propose here a new axiomatic paraconsistent set theory based on the first-order version of some (paraconsistent) LFIs, by admitting that sets, as well as sentences, can be either consistent or inconsistent. A salient feature of such a paraconsistent set theory, inherited from LFIs, is that only consistent and contradictory objects will explode into triviality. Moreover, if we declare that all sets and sentences are consistent, we immediately obtain traditional ZF set theory, and nothing new.”
Logics of formal inconsistencies (LFI) are paraconsistent logics containing the unary connectives $\circ$ and $\bullet$ describing the notions of consistency and inconsistency, respectively. The formula $\circ \varphi$ can be thought of as the formula $\varphi$ is consistent.
In a paraconsistent logic we know that a contradictory pair \( \{ \varphi, \neg \varphi \} \) of sentences are not necessarily explosive. But if \( \circ \varphi \) is included in \( \{ \varphi, \neg \varphi \} \) then the collection will be trivial, i.e.,

\[
\{ \varphi, \neg \varphi, \circ \varphi \} \vdash \psi,
\]

for any formula \( \psi \).
The propositional logic $mbC$ is one of the basic LFIs, having the following axioms with the one inference rule Modus Ponens.

\[
\begin{align*}
\varphi & \rightarrow (\psi \rightarrow \varphi) \\
(\varphi \rightarrow \psi) & \rightarrow (((\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\varphi \rightarrow \gamma)) \\
\varphi & \rightarrow (\psi \rightarrow (\varphi \land \gamma)) \\
(\varphi \land \psi) & \rightarrow \varphi \\
(\varphi \land \psi) & \rightarrow \psi \\
\varphi & \rightarrow (\varphi \lor \psi) \\
\psi & \rightarrow (\varphi \lor \psi) \\
(\phi \rightarrow \gamma) & \rightarrow (((\psi \rightarrow \gamma) \rightarrow (\varphi \lor \psi \rightarrow \gamma)) \\
\varphi \lor (\varphi \rightarrow \psi) & \\
\varphi \lor \neg \varphi & \\
\circ \varphi & \rightarrow (\varphi \rightarrow (\neg \varphi \rightarrow \psi))
\end{align*}
\]
QmbC is the first order predicate extension of mbC by adding the following axioms and inference rules.

Predicate axioms:
\( \varphi[x/t] \rightarrow \exists x \varphi \), if \( t \) is free for \( x \) in \( \varphi \)
\( \forall x \varphi \rightarrow \varphi[x/t] \), if \( t \) is free for \( x \) in \( \varphi \)

If \( \varphi \) is a variant of \( \psi \) then \( \varphi \rightarrow \psi \) is an axiom.

Inference rules:
\( \varphi \rightarrow \psi / \varphi \rightarrow \forall x \psi \), if \( x \) is not free in \( \varphi \)
\( \varphi \rightarrow \psi / \exists x \varphi \rightarrow \psi \), if \( x \) is not free in \( \psi \).
The weak negation $\neg$ in the axiom system of $QmbC$ is not classical. Whereas the strong classical negation $\sim$ can be defined using $\neg$ and $\circ$ as follows.

For any formula $\alpha$ define $\bot_{\alpha} = \alpha \land \neg \alpha \land \circ \alpha$. Then we define

$$\sim \varphi = \varphi \rightarrow \bot_p,$$

where $p$ is the formula $\forall x \ (x = x)$.
The set theory $ZFmbC$ consists of the first order logic $QmbC$ containing two binary predicates ‘$=$’ (for equality) and ‘$\in$’ (for membership), and a unary predicate $C$ (for consistency of sets), together with the following axioms, organized in five groups:

1) The Leibniz axiom for equality: $(x = y) \rightarrow (\varphi \rightarrow \varphi^x_y)$.

2) All the axioms of $ZF^-$. 

3) The Regularity Axiom: 

$$C(x) \rightarrow (\exists y(y \in x) \rightarrow \exists y(y \in x \land \sim \exists z(z \in x \land z \in y))).$$

4) The Unextensionality Axiom:

$$(x \neq y) \leftrightarrow \exists z((z \in x) \land (z \notin y)) \lor \exists z((z \in y) \land (z \notin x)).$$
5) Axioms for the consistency predicate:

- $\forall x \ (C(x) \rightarrow o(x = x))$
- $\forall x \ (\neg o \ (x = x) \rightarrow \neg C(x))$
- $\forall x \ (x \in y \rightarrow C(x)) \rightarrow C(y)$
Some of the results in $ZFmbC$:

- *Separation axiom* can be derived in $ZFmbC$.
- $\vdash \forall x ((x \in x) \land \neg(x \in x) \rightarrow \neg(x = x))$.
- $\vdash \forall x (C(x) \rightarrow \sim(x \in x))$.
- $C(x), x = x, \neg(x = x) \vdash \varphi$, for any formula $\varphi$.
- $C(x), x \in x, \neg(x \in x) \vdash \varphi$, for any formula $\varphi$.

**Theorem.** If $ZF$ is consistent then $ZFmbC$ is non-trivial, i.e., every formula cannot be derived from $ZFmbC$. 
The author of this paper, Zach Weber used a fist order logic TLQ for the base logic to form a paraconsistent set theory. To introduce the set theoretic axioms the author said the following: “Our first-order formal language is now augmented with a variable binding term forming operator \( \{ : - \} \); it remains open how to conservatively add term-forming symbols in relevance contexts, and is not a problem addressed here.”
The set concept is now characterized by two axioms.

**Abstraction:** \( x \in \{ z : \varphi(z) \} \iff \varphi(x). \)

**Extensionality:** \( \forall z (z \in x \iff z \in y) \iff (x = y). \)
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**Extensionality:** \( \forall z (z \in x \iff z \in y) \iff (x = y). \)

**Theorem.** (Comprehension): For any formula \( \varphi(x) \) having one free variable \( x \), \( \vdash \exists y \forall x (x \in y \iff \varphi(x)) \).
Some of the other derived results in this set theory.

- All the axioms of $\mathbb{ZF}^-$ can be proved.
- $\vdash \exists x \ (x \neq x)$.
- $\vdash \exists x \ (x \in a \land x \notin a) \rightarrow a \notin a$.
- There exists a universal set.
- There is a set of all ordinals.
There is a common technique mainly used to build the models of paraconsistent set theory, before the technique of algebra-valued models of some paraconsistent set theory came into existence. Thierry Libert, Olivier Esser, Walter Carnielli, Marcelo Coniglio, etc. have given models of some paraconsistent set theories using the above mentioned technique.
The technique of building the model

Consider a set $M$. Suppose $P(P(M)) := \{ (A, B) : A \cup B = M \}$. A structure $M$ for a paraconsistent set theory is defined by a non-empty set $M$ together with a function $\cdot M$ from $M$ into $P(P(M))$, which associates any $a \in M$ simultaneously to its positive extension $a^+_M$ and its negative extension $a^-_M$, i.e., $\cdot M(a) = (a^+_M, a^-_M)$.  

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- A structure $\mathcal{M}$ for a paraconsistent set theory is defined by a non-empty set $M$ together with a function $[.]_\mathcal{M}$ from $M$ into $\mathcal{P}_P(M)$, which associate any $a \in M$ simultaneously to its positive extension $[a]_\mathcal{M}^+$ and its negative extension $[a]_\mathcal{M}^-$, i.e.,

$$[a]_\mathcal{M} = ([a]_\mathcal{M}^+, [a]_\mathcal{M}^-).$$
Note that there may exists $a \in M$ such that $[a]^+_M \cap [a]^-_M$ is non-empty.
- Note that there may exists $a \in M$ such that $[a]^+_M \cap [a]^-_M$ is non-empty.

- By setting $a \in_M b$ iff $a \in [b]^+_M$ and $a \notin_M b$ iff $a \in [b]^-_M$, for any $a, b \in M$, the structure $\mathcal{M}$ can equivalently be defined as

\[ \mathcal{M} := \langle M, \in_M, \notin_M \rangle \text{ where } \in_M \cup \notin_M = M \times M. \]
Accordingly, ‘$x \in y$’ can be interpreted as being both ‘true’ and ‘false’ for some $x, y$ in $M$. To formalize this, we define the truth function $\epsilon_M$ of the membership relation ‘$\in$’ in $M$ as follows:
Accordingly, ‘\(x \in y\)’ can be interpreted as being both ‘true’ and ‘false’ for some \(x, y\) in \(M\). To formalize this, we define the truth function \(\epsilon_M\) of the membership relation ‘\(\in\)’ in \(M\) as follows:

\[
t \in \epsilon_M(a, b) \text{ iff } a \in_M b
\]
\[
f \in \epsilon_M(a, b) \text{ iff } a \notin_M b,
\]
for all \(a, b \in M\).
Accordingly, ‘$x \in y$’ can be interpreted as being both ‘true’ and ‘false’ for some $x, y$ in $M$. To formalize this, we define the truth function $\epsilon_M$ of the membership relation ‘$\in$’ in $M$ as follows:

- $t \in \epsilon_M(a, b)$ iff $a \in_M b$
- $f \in \epsilon_M(a, b)$ iff $a \not\in_M b$,

for all $a, b \in M$.

So, in this way, $\epsilon_M(a, b)$ takes exactly one of the following truth values:

$$0 := \{f\}, \quad 1 := \{t\}, \quad i := \{t, f\}.$$
In this way a structure for a paraconsistent set theory appears as $\mathcal{M} := \langle M, \epsilon_M \rangle$. 
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The truth degree of the atomic formula ‘$x \in y$’ in a given structure has been defined. More generally, the truth degree of any formula $\varphi$ interpreted within a given structure $\mathcal{M}$ is denoted by $|\varphi|_\mathcal{M}$. Incidentally, whenever we write $|\varphi|_\mathcal{M}$, it will be assumed that an assignment has been given to the free variables of $\varphi$ into $\mathcal{M}$ so that the truth degree of $\varphi$ in $\mathcal{M}$ is computable, inductively.
We are now in a position to define the satisfaction relation $\models$. Consider a structure $\mathcal{M} := \langle M, \epsilon_M \rangle$ and a formula $\varphi$ from the set theoretic language corresponding to $\mathcal{M}$, define

$$\mathcal{M} \models \varphi \text{ iff } t \in |\varphi|_\mathcal{M}.$$
Axioms of Zermelo-Fraenkel set theory

\[\forall x \forall y [\forall z (z \in x \iff z \in y) \rightarrow x = y]\]  
(Extensionality)

\[\forall x \forall y \exists z \forall w (w \in z \iff (w = x \lor w = y))\]  
(Pairing)

\[\exists x [\exists y (\forall z (z \notin y) \land y \in x) \land \forall w \in x \exists u \in x (w \in u)]\]  
(Infinity)

\[\forall x \exists y \forall z (z \in y \iff \exists w \in x (z \in x))\]  
(Union)

\[\forall x \exists y \forall z (z \in y \iff \forall w \in z (w \in x))\]  
(Power Set)
Introduction

Paraconsistent Logic and set theories

Axioms of ZF set theory

Boolean and Heyting valued models

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Ordinals in \(V^{(PS_3)}\)

Non-Classical Behaviour of \(V^{(PS_3)}\)

Conclusion

\[\forall p_0 \cdots \forall p_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \varphi(z, p_0, \ldots, p_n))\]

(Separation\(\varphi\))

\[\forall p_0 \cdots \forall p_{n-1} \forall x [\forall y \in x \exists z \varphi(y, z, p_0, \ldots, p_{n-1})
\rightarrow \exists w \forall v \in x \exists u \in w \varphi(v, u, p_0, \ldots, p_{n-1})]\]

(Replacement\(\varphi\))

\[\forall p_0 \cdots \forall p_n \forall x [\forall y \in x \varphi(y, p_0, \ldots, p_n) \rightarrow \varphi(x, p_0, \ldots, p_n)]
\rightarrow \forall z \varphi(z, p_0, \ldots, p_n)\]

(Foundation\(\varphi\))
Construction of Boolean Valued Model

In the following steps it will be discussed briefly that how a Boolean valued model is constructed and in which sense it becomes a model of ZFC. The whole construction will take place over the *standard model* \( \mathcal{V} \) of ZFC.
Construction of Boolean Valued Model

In the following steps it will be discussed briefly that how a Boolean valued model is constructed and in which sense it becomes a model of ZFC. The whole construction will take place over the *standard model* $V$ of ZFC.

1. Let us take a complete Boolean algebra,
   $$\mathbb{B} = \langle B, \land, \lor, \Rightarrow, *, 0, 1 \rangle.$$
Construction of Boolean Valued Model

In the following steps it will be discussed briefly that how a Boolean valued model is constructed and in which sense it becomes a model of ZFC. The whole construction will take place over the \textit{standard model} $\mathcal{V}$ of ZFC.

1. Let us take a complete Boolean algebra,
   \[ \mathbb{B} = \langle B, \land, \lor, \rightarrow, *, 0, 1 \rangle. \]
2. For any ordinal $\alpha$ we define,
   \[ V_{\alpha}^{(\mathbb{B})} = \{ x : \text{Func}(x) \land \text{ran}(x) \subseteq B \land \exists \xi < \alpha (\text{dom}(x) \subseteq V_{\xi}^{(\mathbb{B})}) \} \]
Construction of Boolean Valued Model

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1. Let us take a complete Boolean algebra, $\mathbb{B} = \langle B, \land, \lor, \rightarrow, *, 0, 1 \rangle$.

2. For any ordinal $\alpha$ we define,

$$V^{(\mathbb{B})}_\alpha = \{ x : \text{Func}(x) \land \text{ran}(x) \subseteq B \land \exists \xi < \alpha (\text{dom}(x) \subseteq V^{(\mathbb{B})}_\xi) \}$$

3. Using the above we get a Boolean valued model as,

$$V^{(\mathbb{B})} = \{ x : \exists \alpha (x \in V^{(\mathbb{B})}_\alpha) \}$$
4 Extend the language of classical ZFC by adding a name corresponding to each element of $\mathbf{V}(\mathbb{B})$, in it.

5 Associate every formula of the extended language with a value of $B$ by the map $[.]$. First we give the algebraic expressions which associate the two basic well-formed formulas with values of $B$. For any $u, v$ in $\mathbf{V}(\mathbb{B})$,

$$\llbracket u \in v \rrbracket = \bigvee_{x \in \text{dom}(v)} (v(x) \land \llbracket x = u \rrbracket)$$

$$\llbracket u = v \rrbracket = \bigwedge_{x \in \text{dom}(u)} (u(x) \Rightarrow \llbracket x \in v \rrbracket) \land \bigwedge_{y \in \text{dom}(v)} (v(y) \Rightarrow \llbracket y \in u \rrbracket)$$
Then for any sentences $\sigma$ and $\tau$ of the new language we define,

\[
\begin{align*}
[\sigma \land \tau] &= [\sigma] \land [\tau] \\
[\sigma \lor \tau] &= [\sigma] \lor [\tau] \\
[\sigma \rightarrow \tau] &= [\sigma] \Rightarrow [\tau] \\
[\neg \sigma] &= [\sigma]^* \\
[\forall x \varphi(x)] &= \bigwedge_{x \in V^{(B)}} [\varphi(x)] \\
[\exists x \varphi(x)] &= \bigvee_{x \in V^{(B)}} [\varphi(x)]
\end{align*}
\]

A sentence $\sigma$ will be called valid in $V^{(B)}$ or $V^{(B)}$ will be called a model of a sentence $\sigma$ if $[\sigma] = 1$. It will be denoted as $V^{(B)} \models \sigma$. 
Then we ultimately get the following celebrated result:

**Theorem**

*For any complete Boolean algebra* \( \mathcal{B} \), \( V(\mathcal{B}) \mod VFC \), *i.e.*, all the classical logic axioms and ZFC axioms are valid in* \( V(\mathcal{B}) \).
Instead of a complete Boolean algebra if we take a complete Heyting algebra $\mathbb{H}$ by the similar construction one can conclude

**Theorem**

$V^{(H)} \models IZF$, where $IZF$ stands for the intuitionistic Zermelo-Fraenkel set theory, a set theory based on the intuitionistic logic.
Bounded Quantification Property

The following property $BQ_\varphi$ named after the bounded quantification property for the formula $\varphi$, played a very important role in proving the above two theorems.

$$\left[ \forall x \in u \varphi(x) \right] = \bigwedge_{x \in \text{dom}(u)} \left( u(x) \Rightarrow [\varphi(x)] \right) \quad (BQ_\varphi)$$

where $\forall x \in u \varphi(x)$ is the abbreviation for $\forall x (x \in u \rightarrow \varphi(x))$. 
Bounded Quantification Property

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\[
\left[ \forall x \in u \varphi(x) \right] = \bigwedge_{x \in \text{dom}(u)} (u(x) \Rightarrow \left[ \varphi(x) \right]) \quad (\text{BQ}_\varphi)
\]

where $\forall x \in u \varphi(x)$ is the abbreviation for $\forall x(x \in u \rightarrow \varphi(x))$.

It can be proved that for any complete Boolean algebra $B$ and complete Heyting algebra $H$ the property $\text{BQ}_\varphi$ holds in both $V(B)$ and $V(H)$ for all formula $\varphi$. 
Reasonable Implication Algebra

Definition

An algebra $\mathbb{A} = \langle A, \land, \lor, 1, 0, \Rightarrow \rangle$ is called a reasonable implication algebra if $\langle A, \land, \lor, 1, 0 \rangle$ is a complete distributive lattice and $\Rightarrow$ has the following properties:
Reasonable Implication Algebra

**Definition**

An algebra $\mathbb{A} = \langle A, \wedge, \vee, 1, 0, \Rightarrow \rangle$ is called a reasonable implication algebra if $\langle A, \wedge, \vee, 1, 0 \rangle$ is a complete distributive lattice and $\Rightarrow$ has the following properties:

**P1:** $x \wedge y \leq z$ implies $x \leq y \Rightarrow z$.

**P2:** $y \leq z$ implies $x \Rightarrow y \leq x \Rightarrow z$.

**P3:** $y \leq z$ implies $z \Rightarrow x \leq y \Rightarrow x$. 
Definition

An algebra \( A = \langle A, \wedge, \vee, 1, 0, \Rightarrow \rangle \) is called a reasonable implication algebra if \( \langle A, \wedge, \vee, 1, 0 \rangle \) is a complete distributive lattice and \( \Rightarrow \) has the following properties:

**P1:** \( x \wedge y \leq z \) implies \( x \leq y \Rightarrow z \).

**P2:** \( y \leq z \) implies \( x \Rightarrow y \leq x \Rightarrow z \).

**P3:** \( y \leq z \) implies \( z \Rightarrow x \leq y \Rightarrow x \).

A reasonable implication algebra is said to be *deductive* if it satisfies

\[
(x \wedge y) \Rightarrow z = x \Rightarrow (y \Rightarrow z).
\]
Negation Free Formulas, NFF

If $\mathcal{L}$ is any first-order language including the connectives $\land$, $\lor$, $\bot$ and $\rightarrow$ and $\Lambda$ any class of $\mathcal{L}$-formulas, we denote closure of $\Lambda$ under $\land$, $\lor$, $\bot$, $\exists$, $\forall$, and $\rightarrow$ by $\text{Cl}(\Lambda)$ and call it the *negation-free closure of $\Lambda$*. A class $\Lambda$ of formulas is *negation-free closed* if $\text{Cl}(\Lambda) = \Lambda$. By NFF we denote the *negation-free closure of the atomic formulas*; its elements are called the *negation-free formulas*. 
An Algebra-Valued Model of a Set Theory

Following the above mentioned constructions we have proved:

**Theorem**

*Let \( A \) be a deductive reasonable implication algebra such that \( V(A) \) satisfies the NFF-bounded quantification property (NFF – BQ\( \varphi \)). Then Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in \( V(A) \).*

NFF-(...) stands for the instances of (...) only for the negation free formulas.

Is there any algebra other than complete Boolean algebras and complete Heyting algebras which is a deductive reasonable implication algebra satisfying the NFF – BQ$_{\varphi}$?
Three Valued Matrix $\text{PS}_3$

Let us consider the three valued matrix $\text{PS}_3 = \langle \{1, 1/2, 0\}, \wedge, \vee, \implies, *, 1, 0 \rangle$ where the truth tables for the operators are given below:
Three Valued Matrix $\mathbf{PS}_3$

Let us consider the three valued matrix

$\mathbf{PS}_3 = \langle \{1, \frac{1}{2}, 0\}, \land, \lor, \rightarrow, *, 1, 0 \rangle$ where the truth tables for the operators are given below:

$$
\begin{array}{c|ccc}
\land & 1 & \frac{1}{2} & 0 \\
\hline
1 & 1 & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{c|ccc}
\lor & 1 & \frac{1}{2} & 0 \\
\hline
1 & 1 & 1 & 1 \\
\frac{1}{2} & 1 & 1 & \frac{1}{2} \\
0 & 1 & \frac{1}{2} & 0 \\
\end{array}
$$

$$
\begin{array}{c|ccc}
\rightarrow & 1 & \frac{1}{2} & 0 \\
\hline
1 & 1 & 1 & 0 \\
\frac{1}{2} & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
\end{array}
$$

$$
\begin{array}{c|c}
* & \\
\hline
1 & 0 \\
\frac{1}{2} & \frac{1}{2} \\
0 & 1 \\
\end{array}
$$

and $\{1, \frac{1}{2}\}$ is taken as the designated set.
The Answer

It can be verified that $PS_3$ is a deductive reasonable implication algebra and $BQ_{\varphi}$ holds in $V^{(PS_3)}$ for all negation free formula $\varphi$. 
So we can conclude that Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in $V^{(PS_3)}$. 
So we can conclude that Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in $V^{(P_{S3})}$.

Moreover it is proved separately that NFF-Regularity is also valid in $V^{(P_{S3})}$. 
Logic Corresponding to $\mathsf{PS}_3$

What is the logic corresponding to $\mathsf{PS}_3$? Is it paraconsistent?
What is the logic corresponding to $\text{PS}_3$? Is it paraconsistent?

We have found a logic $\text{LPS}_3$ which is *sound* and *complete* with respect to $\text{PS}_3$. More interestingly it can be proved that $\text{LPS}_3$ is a paraconsistent logic.
Axioms for LPS₃

The following formulas are taken as the axioms for LPS₃:

(Ax1) \( \varphi \rightarrow (\psi \rightarrow \varphi) \)
(Ax2) \( (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \)
(Ax3) \( \varphi \land \psi \rightarrow \varphi \)
(Ax4) \( \varphi \land \psi \rightarrow \psi \)
(Ax5) \( \varphi \rightarrow \varphi \lor \psi \)
(Ax6) \( (\varphi \rightarrow \gamma) \land (\psi \rightarrow \gamma) \rightarrow (\varphi \lor \psi \rightarrow \gamma) \)
(Ax7) \( (\varphi \rightarrow \psi) \land (\varphi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \psi \land \gamma) \)
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Matrix of $\mathbb{PS}_3$

$\mathbb{PS}_3$-valued model

The Logic $L_{\mathbb{PS}_3}$

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Paraconsistent Set Theory

(Ax8) $\varphi \leftrightarrow \neg\neg\varphi$

(Ax9) $\neg(\varphi \land \psi) \leftrightarrow (\neg\varphi \lor \neg\psi)$

(Ax10) $(\varphi \land \neg\varphi) \rightarrow (\neg(\psi \rightarrow \varphi) \rightarrow \gamma)$

(Ax11) $(\varphi \rightarrow \psi) \rightarrow (\neg(\varphi \rightarrow \gamma) \rightarrow \psi)$

(Ax12) $(\neg\varphi \rightarrow \psi) \rightarrow (\neg(\gamma \rightarrow \varphi) \rightarrow \psi)$

(Ax13) $\bot \rightarrow \varphi$

(Ax14) $(\varphi \land (\psi \rightarrow \bot)) \rightarrow \neg(\varphi \rightarrow \psi)$

(Ax15) $(\varphi \land (\neg\varphi \rightarrow \bot)) \lor (\varphi \land \neg\varphi) \lor (\neg\varphi \land (\varphi \rightarrow \bot))$

where $\varphi, \psi, \gamma$ are any well formed formulas and $\bot$ is the abbreviation for $\neg(\theta \rightarrow \theta)$ for any arbitrary formula $\theta$. 
Rules for $LPS_3$

The rules for $LPS_3$ are the following:

1. \[ \varphi, \psi \quad \frac{\varphi \land \psi}{\varphi \land \psi} \]
2. \[ \varphi, \varphi \rightarrow \psi \quad \frac{\varphi}{\psi} \]
Soundness and completeness of $LPS_3$

Let $\vdash$ and $\models$ be the syntactic and semantic consequence relations respectively defined in the usual way with respect to the above mentioned axiom system and the matrix $PS_3$. 
Soundness and completeness of $LPS_3$

Let $\vdash$ and $\models$ be the syntactic and semantic consequence relations respectively defined in the usual way with respect to the above mentioned axiom system and the matrix $PS_3$.

**Theorem**

*Soundness: For any formula $\varphi$ and a set of formulas $\Gamma$, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.***

**Theorem**

*Completeness: For any formula $\varphi$, if $\models \varphi$ then $\vdash \varphi$.***

Hence we have reached to the fact that

\( V^{(PS_3)} \) is an algebra-valued model of a paraconsistent set theory.
Some Classical Definitions in Metalanguage

**Definition**

A set \( x \) is said to be transitive if every element of \( x \) is a subset of \( x \), or equivalently, if \( y \in z \) and \( z \in x \) implies \( y \in x \).
Some Classical Definitions in Metalanguage

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A set $A$ is said to be well-ordered by a relation $R$ if $R$ is a linear order on $A$ and any non-empty subset of $A$ has a least element with respect to $R$. 
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**Definition**
An ordinal number is a transitive set well-ordered by $\in$. 

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For each $\alpha \in \text{ORD}$ the $\alpha$-like names in $V^{(PS_3)}$ are defined by transfinite recursion as follows.

**Definition**

An element $x \in V^{(PS_3)}$ is called

1. **0-like** if for every $y \in \text{dom}(x)$, we have that $x(y) = 0$; and
2. **$\alpha$-like** if for each $\beta \in \alpha$ there exists $y \in \text{dom}(x)$ which is $\beta$-like and $x(y) \in \{1, 1/2\}$, and for any $z \in \text{dom}(x)$ if it is not $\beta$-like for any $\beta \in \alpha$ then $x(z) = 0$. 
\( \alpha \)-like Elements Meets Our Expectations

**Theorem**

Let \( x \in V^{(PS_3)} \) be \( \alpha \)-like for some \( \alpha \in \text{ORD} \). For any \( y \in V^{(PS_3)} \),

\[
[x = y] = 1 \text{ if and only if } y \text{ is } \alpha \text{-like.}
\]
α-like Elements Meets Our Expectations

**Theorem**

Let $x \in \mathbf{V}^{(\text{PS}_3)}$ be $\alpha$-like for some $\alpha \in \text{ORD}$. For any $y \in \mathbf{V}^{(\text{PS}_3)}$, $[x = y] = 1$ if and only if $y$ is $\alpha$-like.

**Theorem**

Let $x \in \mathbf{V}^{(\text{PS}_3)}$ be $\alpha$-like for some non-zero $\alpha \in \text{ORD}$. For any $y \in \mathbf{V}^{(\text{PS}_3)}$, $[y \in x] \in \{1, 1/2\}$ if and only if $y$ is $\beta$-like for some $\beta \in \alpha$. 
Ordinals in First Order Language

As promised earlier, the definitions of transitive set, linear-ordered set, well-ordered set and ordinal number is written below in the set theoretic language.
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\text{Trans}(x) = \forall y \forall z (z \in y \land y \in x \rightarrow z \in x)
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\[ \text{WO}_\in(x) = \text{LO}(x) \land \forall y (y \subseteq x \land \neg (y = \emptyset) \rightarrow \exists z (z \in y \land z \cap y = \emptyset)) \]
As promised earlier, the definitions of transitive set, linear-ordered set, well-ordered set and ordinal number is written below in the set theoretic language.

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\[
\text{ORD}(x) = \text{Trans}(x) \land \text{WO}_{\in}(x)
\]
Ordinals in First Order Language

The following abbreviations are used in $WO_{\in}(x)$:

$$y \subseteq x := \forall t (t \in y \rightarrow t \in x),$$

$$\neg (y = \emptyset) := \exists z (z \in y),$$

$$(z \cap y = \emptyset) := \neg \exists w (w \in z \land w \in y).$$
Finally, we can connect the notion of $\alpha$-like name to the set theoretic notion of ordinals:

**Lemma**

Let $\alpha \in \text{ORD}$ and $u$ be an $\alpha$-like element in $V^{(PS_3)}$. Then the following hold:

1. $V^{(PS_3)} \models \text{Trans}(u)$
2. $V^{(PS_3)} \models \text{LO}(u)$
3. $V^{(PS_3)} \models \text{WO}_\epsilon(u)$
Some Results on Ordinal-Like Elements

Hence we conclude the following theorem:

**Theorem**

Let \( \alpha \in \text{ORD} \) and \( u \) be an \( \alpha \)-like element in \( \mathcal{V}(PS_3) \). Then \( \mathcal{V}(PS_3) \models \text{ORD}(u) \).
Some Results on Ordinal-Like Elements

Hence we conclude the following theorem:

**Theorem**

Let $\alpha \in \text{ORD}$ and $u$ be an $\alpha$-like element in $V^{(PS_3)}$. Then $V^{(PS_3)} \models \text{ORD}(u)$.

Like the classical set theory we also have

**Theorem**

*There is no set of all ordinals:*

$$V^{(PS_3)} \not\models \exists O \ \forall x (\text{ORD}(x) \rightarrow x \in O).$$

Leibniz’s Law of the Indiscernibility of Identicals

It can be proved that Leibniz’s law of the indiscernibility of identicals

\[ \forall x \forall y (x = y \land \varphi(x) \rightarrow \varphi(y)) \]

is not valid in \( V^{(PS_3)} \) for all formula \( \varphi \).
Leibniz’s Law of the Indiscernibility of Identicals

It can be proved that Leibniz’s law of the indiscernibility of identicals

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is not valid in \( V(PS_3) \) for all formula \( \varphi \).

On the other hand it is also proved that for any instantiations of Leibniz’s law with NFF-formulas \( \varphi \) is valid.
Can we identify a formula $\varphi$ in the language of set theory so that both $\varphi$ and $\neg \varphi$ are true in $V^{(PS_3)}$?
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Let $\varphi := \exists x \exists y \exists z (x = y \land z \in x \land z \notin y)$. Then it can be proved that $\llbracket \varphi \rrbracket = \frac{1}{2}$ and hence $\llbracket \neg \varphi \rrbracket = \frac{1}{2}^* = \frac{1}{2}$ which leads to the fact that both $\varphi$ and $\neg \varphi$ are valid in $V^{(PS_3)}$. 
We have found an algebra, \( A \) called deductive reasonable implication algebra.

Proved that \( V(A) \) is an algebra valued model of the set theory \( NFF - ZF \).

Found a 3-valued matrix \( PS_3 \) which is neither Boolean nor Heyting but a deductive reasonable implication algebra.

The logic \( LPS_3 \) which is sound and complete with respect to \( PS_3 \) is a paraconsistent logic.

As a consequence, \( V(PS_3) \) is a model of some paraconsistent set theory.

Defined ordinal-like elements inside \( V(PS_3) \) and studied some classical and non-classical properties of them.

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Paraconsistent Set Theory
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Paraconsistent Set Theory
Thank You