Paraconsistent Set Theory

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In the context of classical logic the above two notions are equivalent.

A logic is said to be paraconsistent if there exists a set Γ of formulas such that Γ is inconsistent but not explosive.

Definition

To explain it simply we have the following definition of paraconsistent logic:

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A logic is called paraconsistent if there exist formulas φ and ψ such that $\{\varphi, \neg\varphi\} \nvdash \psi$.

Some Well Known Paraconsistent Logics

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For example: Jaskowśki's paraconsistent logic, Da Costa's paraconsistent logic systems C_n where $0 < n < \omega$, Priest's logic of paradox, also other paraconsistent logics made by C. Mortensen, R. Brady, J. Marcos, W.A. Carnielli, A. Avron etc.

Paraconsistent Set Theories- a Survey

Syntactic developments of some paraconsistent set theories are already made by the paraconsistentists like N. C. A. da Costa, Zach Weber, Walter Carnielli, Marcelo E. Coniglio etc. Whereas the semantical developments of this area are not strongly established other than some research work done by Thierry Libert, Olivier Esser etc.

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Paraconsistent set theory by predicating on consistency

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Paraconsistent set theory by predicating on consistency

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"We propose here a new axiomatic paraconsistent set theory based on the first-order version of some (paraconsistent) LFIs, by admitting that sets, as well as sentences, can be either consistent or inconsistent. A salient feature of such a paraconsistent set theory, inherited from LFIs, is that only consistent and contradictory objects will explode into triviality. Moreover, if we declare that all sets and sentences are consistent, we immediately obtain traditional ZF set theory, and nothing new."

cont.

Logics of formal inconsistencies (LFI) are paraconsistent logics containing the unary connectives $\circ$ and $\bullet$ describing the notions of consistency and inconsistency, respectively. The formula $\circ\varphi$ can be thought of as the formula φ is consistent.

cont.

In a paraconsistent logic we know that a contradictory pair $\{\varphi, \neg\varphi\}$ of sentences are not necessarily explosive. But if $\circ\varphi$ is included in $\{\varphi, \neg\varphi\}$ then the collection will be trivial, i.e.,

$$\{\varphi, \neg\varphi, \circ\varphi\} \vdash \psi,$$

for any formula ψ.

cont.

The propositional logic *mbC* is one of the basic LFIs, having the following axioms with the one inference rule Modus Ponens.

$\varphi \to (\psi \to \varphi)$

$(\varphi \to \psi) \to ((\varphi \to (\psi \to \gamma)) \to (\varphi \to \gamma)$

$\varphi \to (\psi \to (\varphi \wedge \gamma))$

$(\varphi \wedge \psi) \to \varphi$

$(\varphi \wedge \psi) \to \psi$

$\varphi \to (\varphi \vee \psi)$

$\psi \to (\varphi \vee \psi)$

$(\phi \to \gamma) \to ((\psi \to \gamma) \to (\varphi \vee \psi \to \gamma))$

$\varphi \vee (\varphi \to \psi)$

$\varphi \vee \neg\varphi$

$\circ\varphi \to (\varphi \to (\neg\varphi \to \psi))$

cont.

QmbC is the first order predicate extension of *mbC* by adding the following axioms and inference rules.

Predicate axioms:

$\varphi[x/t] \to \exists x\ \varphi$, if t is free for x in φ

$\forall x\ \varphi \to \varphi[x/t]$, if t is free for x in φ

If φ is a variant of ψ then $\varphi \to \psi$ is an axiom.

Inference rules:

$\varphi \to \psi / \varphi \to \forall x\ \psi$, if x is not free in φ

$\varphi \to \psi / \exists x\ \varphi \to \psi$, if x is not free in ψ.

cont.

The weak negation $\neg$ in the axiom system of *QmbC* is not classical. Whereas the strong classical negation $\sim$ can be defined using $\neg$ and $\circ$ as follows.
For any formula α define $\bot_\alpha = \alpha \wedge \neg\alpha \wedge \circ\alpha$. Then we define

$$\sim \varphi = \varphi \to \bot_p,$$

where p is the formula $\forall x\,(x = x)$.

cont.

The set theory *ZFmbC* consists of the first order logic *QmbC* containing two binary predicates '$=$' (for equality) and '$\in$' (for membership), and a unary predicate C (for consistency of sets), together with the following axioms, organized in five groups:
1) The Leibniz axiom for equality: $(x = y) \to (\varphi \to \varphi^x_y)$.
2) All the axioms of ZF^-.
3) The Regularity Axiom:
$\mathrm{C}(x) \to (\exists y(y \in x) \to \exists y(y \in x \wedge \sim \exists z(z \in x \wedge z \in y)))$.
4) The Unextensionality Axiom:
$(x \neq y) \leftrightarrow \exists z((z \in x) \wedge (z \notin y)) \vee \exists z((z \in y) \wedge (z \notin x))$.

cont.

5) Axioms for the consistency predicate:

- $\forall x\,(\mathrm{C}(x) \rightarrow \circ(x = x))$
- $\forall x\,(\neg \circ (x = x) \rightarrow \neg\mathrm{C}(x))$
- $\forall x\,(x \in y \rightarrow \mathrm{C}(x)) \rightarrow \mathrm{C}(y))$

cont.

Some of the results in *ZFmbC*:

- *Seperation axiom* can be derived in *ZFmbC*.
- $\vdash \forall x\,((x \in x) \wedge \neg(x \in x) \rightarrow \neg(x = x))$.
- $\vdash \forall x\,(\mathrm{C}(x) \rightarrow \sim (x \in x))$.
- $\mathrm{C}(x), x = x, \neg(x = x) \vdash \varphi$, for any formula φ.
- $\mathrm{C}(x), x \in x, \neg(x \in x) \vdash \varphi$, for any formula φ.
- **Theorem.** If ZF is consistent then *ZFmbC* is non-trivial, i.e., every formula cannot be derived from *ZFmbC*.

Transfinite numbers in paraconsistent set theory

The author of this paper, Zach Weber used a fist order logic TLQ for the base logic to form a paraconsistent set theory. To introduce the set theoretic axioms the author said the following:
"Our first-order formal language is now augmented with a variable binding term forming operator $\{. : -\}$; it remains open how to conservatively add term-forming symbols in relevance contexts, and is not a problem addressed here."

cont.

The set concept is now characterized by two axioms.

Abstraction: $x \in \{z : \varphi(z)\} \leftrightarrow \varphi(x)$.

Extensionality: $\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow (x = y)$.

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Theorem. (Cmprehension):For any formula $\varphi(x)$ having one free variable x, $\vdash \exists y \forall x(x \in y \leftrightarrow \varphi(x))$.

cont.

Some of the other derived results in this set theory.

- All the axioms of ZF^- can be proved.
- $\vdash \exists x\ (x \neq x)$.
- $\vdash \exists x(x \in a \wedge x \notin a) \to a \notin a$.
- There exists a universal set.
- There is a set of all ordinals.

Some models of paraconsistent set theory

There is a common technique mainly used to build the models of paraconsistent set theory, before the technique of algebra-valued models of some paraconsistent set theory came into existence. Thierry Libert, Olivier Esser, Walter Carnielli, Marcelo Coniglio, etc. have given models of some paraconsistent set theories using the above mentioned technique.

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- Suppose $\mathcal{P}_P(\mathrm{M}) := \{(A, B) : A \cup B = \mathrm{M}\}$.
- A structure $\mathcal{M}$ for a paraconsistent set theory is defined by a non-empty set M together with a function $[.]_{\mathcal{M}}$ from M into $\mathcal{P}_P(\mathrm{M})$, which associate any $a \in \mathrm{M}$ simultaneously to its positive extension $[a]^+_{\mathcal{M}}$ and its negative extension $[a]^-_{\mathcal{M}}$, i.e.,

$$[a]_{\mathcal{M}} = ([a]^+_{\mathcal{M}}, [a]^-_{\mathcal{M}}).$$

Cont.

- Note that there may exists $a \in \mathrm{M}$ such that $[a]^{+}_{\mathcal{M}} \cap [a]^{-}_{\mathcal{M}}$ is non-empty.

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- By setting $a \in_{\mathcal{M}} b$ iff $a \in [b]^{+}_{\mathcal{M}}$ and $a \notin_{\mathcal{M}} b$ iff $a \in [b]^{-}_{\mathcal{M}}$, for any $a, b \in \mathrm{M}$, the structure $\mathcal{M}$ can equivalently be defined as

$$\mathcal{M} := \langle \mathrm{M}, \in_{\mathcal{M}}, \notin_{\mathcal{M}} \rangle \text{ where } \in_{\mathcal{M}} \cup \notin_{\mathcal{M}} = \mathrm{M} \; x \; \mathrm{M}.$$

cont.

- Accordingly, '$x \in y$' can be interpreted as being both 'true' and 'false' for some x, y in M. To formalize this, we define the truth function $\epsilon_{\mathcal{M}}$ of the membership relation '$\in$' in M as follows:

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  $t \in \epsilon_{\mathcal{M}}(a, b)$ iff $a \in_{\mathcal{M}} b$
  $f \in \epsilon_{\mathcal{M}}(a, b)$ iff $a \notin_{\mathcal{M}} b$,
  for all $a, b \in \mathrm{M}$.

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  $f \in \epsilon_{\mathcal{M}}(a, b)$ iff $a \notin_{\mathcal{M}} b$,
  for all $a, b \in \mathrm{M}$.
- So, in this way, $\epsilon_{\mathcal{M}}(a, b)$ takes exactly one of the following *truth values*:

$$0 := \{f\},\ 1 := \{t\},\ i := \{t, f\}.$$

cont.

- In this way a structure for a paraconsistent set theory appears as $\mathcal{M} := \langle \mathrm{M}, \epsilon_{\mathcal{M}} \rangle$.

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- In this way a structure for a paraconsistent set theory appears as $\mathcal{M} := \langle \mathrm{M}, \epsilon_{\mathcal{M}} \rangle$.
- The truth degree of the atomic formula '$x \in y$' in a given structure has been defined. More generally, the truth degree of any formula φ interpreted within a given structure M is denoted by $|\varphi|_{\mathcal{M}}$. Incidentally, whenever we write $|\varphi|_{\mathcal{M}}$, it will be assumed that an assignment has been given to the free variables of φ into M so that the truth degree of φ in M is computable, inductively.

cont.

We are now in a position to define the *satisfaction relation* $\models$. Consider a structure $\mathcal{M} := \langle \mathrm{M}, \epsilon_{\mathcal{M}} \rangle$ and a formula φ from the set theoretic language corresponding to $\mathcal{M}$, define

$$\mathcal{M} \models \varphi \text{ iff } t \in |\varphi|_{\mathcal{M}}.$$

Axioms of Zermelo-Fraenkel set theory

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y] \qquad \text{(Extensionality)}$$

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y)) \qquad \text{(Pairing)}$$

$$\exists x [\exists y (\forall z (z \notin y) \wedge y \in x) \wedge \forall w \in x \exists u \in x (w \in u)] \qquad \text{(Infinity)}$$

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x (z \in x)) \qquad \text{(Union)}$$

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w \in z (w \in x)) \qquad \text{(Power Set)}$$

$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, p_0, \ldots, p_n)) \tag{Separation$_\varphi$}$$

$$\forall p_0 \cdots \forall p_{n-1} \forall x [\forall y \in x \exists z \varphi(y, z, p_0, \ldots, p_{n-1}) \rightarrow \exists w \forall v \in x \exists u \in w \ \varphi(v, u, p_0, \ldots, p_{n-1})] \tag{Replacement$_\varphi$}$$

$$\forall p_0 \cdots \forall p_n \forall x [\forall y \in x \ \varphi(y, p_0, \ldots, p_n) \rightarrow \varphi(x, p_0, \ldots, p_n)] \rightarrow \forall z \varphi(z, p_0, \ldots, p_n) \tag{Foundation$_\varphi$}$$

Construction of Boolean Valued Model

In the following steps it will be discussed briefly that how a Boolean valued model is constructed and in which sense it becomes a model of ZFC. The whole construction will take place over the *standard model* **V** of ZFC.

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2. For any ordinal α we define,

$$\mathbf{V}_\alpha^{(\mathbb{B})} = \{x : \mathrm{Func}(x) \wedge \mathrm{ran}(x) \subseteq B \wedge \exists\xi < \alpha(\mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{(\mathbb{B})})\}$$

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3. Using the above we get a Boolean valued model as,

$$\mathbf{V}^{(\mathbb{B})} = \{x : \exists\alpha(x \in \mathbf{V}_{\alpha}^{(\mathbb{B})})\}$$

4. Extend the language of classical ZFC by adding a name corresponding to each element of $\mathbf{V}^{(\mathbb{B})}$, in it.
5. Associate every formula of the extended language with a value of B by the map $[\![.]\!]$. First we give the algebraic expressions which associate the two basic well-formed formulas with values of B. For any u, v in $\mathbf{V}^{(\mathbb{B})}$,

$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!])$$

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!])$$

6. Then for any sentences σ and τ of the new language we define,

$$\llbracket \sigma \wedge \tau \rrbracket = \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket$$
$$\llbracket \sigma \vee \tau \rrbracket = \llbracket \sigma \rrbracket \vee \llbracket \tau \rrbracket$$
$$\llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \Rightarrow \llbracket \tau \rrbracket$$
$$\llbracket \neg \sigma \rrbracket = \llbracket \sigma \rrbracket^{*}$$
$$\llbracket \forall x \varphi(x) \rrbracket = \bigwedge_{x \in \mathbf{V}^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$
$$\llbracket \exists x \varphi(x) \rrbracket = \bigvee_{x \in \mathbf{V}^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$

7. A sentence σ will be called *valid* in $\mathbf{V}^{(\mathbb{B})}$ or $\mathbf{V}^{(\mathbb{B})}$ will be called a model of a sentence σ if $\llbracket \sigma \rrbracket = 1$. It will be denoted as $\mathbf{V}^{(\mathbb{B})} \models \sigma$.

8. Then we ultimately get the following celebrated result:

Theorem

For any complete Boolean algebra $\mathbb{B}$, $\mathbf{V}^{(\mathbb{B})} \models \mathrm{ZFC}$, i.e., all the classical logic axioms and ZFC axioms are valid in $\mathbf{V}^{(\mathbb{B})}$.

Heyting-Valued Model

Instead of a complete Boolean algebra if we take a complete Heyting algebra $\mathbb{H}$ by the similar construction one can conclude

Theorem

$\mathbf{V}^{(\mathbb{H})} \models \mathrm{IZF}$, *where* IZF *stands for the intuitionistic Zermelo-Fraenkel set theory, a set theory based on the intuitionistic logic.*

Bounded Quantification Property

The following property BQ_φ named after the bounded quantification property for the formula φ, played a very important role in proving the above two theorems.

$$[\![\forall x \in u\varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]) \qquad (\mathrm{BQ}_\varphi)$$

where $\forall x \in u\varphi(x)$ is the abbreviation for $\forall x(x \in u \to \varphi(x))$.

Back to the theorem

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Back to the theorem

It can be proved that for any complete Boolean algebra $\mathbb{B}$ and complete Heyting algebra $\mathbb{H}$ the property BQ_φ holds in both $\mathbf{V}^{(\mathbb{B})}$ and $\mathbf{V}^{(\mathbb{H})}$ for all formula φ.

Reasonable Implication Algebra

Definition

An algebra $\mathbb{A} = \langle A, \wedge, \vee, \mathbf{1}, \mathbf{0}, \Rightarrow \rangle$ is called a reasonable implication algebra if $\langle A, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a complete distributive lattice and $\Rightarrow$ has the following properties:

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P1: $x \wedge y \leq z$ implies $x \leq y \Rightarrow z$.

P2: $y \leq z$ implies $x \Rightarrow y \leq x \Rightarrow z$.

P3: $y \leq z$ implies $z \Rightarrow x \leq y \Rightarrow x$.

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P3: $y \leq z$ implies $z \Rightarrow x \leq y \Rightarrow x$.

A reasonable implication algebra is said to be *deductive* if it satisfies

$$(x \wedge y) \Rightarrow z = x \Rightarrow (y \Rightarrow z).$$

return

Negation Free Formulas, NFF

If $\mathcal{L}$ is any first-order language including the connectives $\wedge$, $\vee$, $\bot$ and $\rightarrow$ and Λ any class of $\mathcal{L}$-formulas, we denote closure of Λ under $\wedge$, $\vee$, $\bot$, $\exists$, $\forall$, and $\rightarrow$ by $\mathrm{Cl}(\Lambda)$ and call it the *negation-free closure of* Λ. A class Λ of formulas is *negation-free closed* if $\mathrm{Cl}(\Lambda) = \Lambda$. By NFF we denote the negation-free closure of the atomic formulas; its elements are called the *negation-free formulas*.

An Algebra-Valued Model of a Set Theory

Following the above mentioned constructions we have proved:

Theorem

Let $\mathbb{A}$ be a deductive reasonable implication algebra such that $\mathbf{V}^{(\mathbb{A})}$ satisfies the NFF*-bounded quantification property* $(\mathrm{NFF} - \mathrm{BQ}_\varphi)$*. Then Extensionality, Pairing, Infinity, Union, Power Set,* NFF*-Separation and* NFF*-Replacement are valid in $\mathbf{V}^{(\mathbb{A})}$.*

NFF-(...) stands for the instances of (...) only for the negation free formulas.

(Löwe, B., and S. Tarafder, Generalized Algebra-Valued Models of Set Theory, *Review of Symbolic Logic*, Cambridge University Press, 8(1), pp. 192–205, 2015.)

BQ_φ
return

Is there any algebra other than complete Boolean algebras and complete Heyting algebras which is a deductive reasonable implication algebra satisfying the $\mathrm{NFF} - \mathrm{BQ}_{\varphi}$?

Three Valued Matrix PS_3

Let us consider the three valued matrix $\mathrm{PS}_3 = \langle \{1, 1/2, 0\}, \wedge, \vee, \Rightarrow, ^*, 1, 0 \rangle$ where the truth tables for the operators are given below:

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$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

$*$	
1	0
1/2	1/2
0	1

and $\{1, 1/2\}$ is taken as the designated set.

The Answer

It can be verified that PS_3 is a deductive reasonable implication algebra and BQ_φ holds in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all negation free formula φ.

So we can conclude that Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

So we can conclude that Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Moreover it is proved separately that NFF-Regularity is also valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Logic Corresponding to PS_3

What is the logic corresponding to PS_3? Is it paraconsistent?

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We have found a logic LPS_3 which is *sound* and *complete* with respect to PS_3. More interestingly it can be proved that LPS_3 is a paraconsistent logic.

Axioms for LPS_3

The following formulas are taken as the axioms for LPS_3:

$$
\begin{array}{ll}
(\mathrm{Ax1}) & \varphi \to (\psi \to \varphi) \\
(\mathrm{Ax2}) & (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma)) \\
(\mathrm{Ax3}) & \varphi \wedge \psi \to \varphi \\
(\mathrm{Ax4}) & \varphi \wedge \psi \to \psi \\
(\mathrm{Ax5}) & \varphi \to \varphi \vee \psi \\
(\mathrm{Ax6}) & (\varphi \to \gamma) \wedge (\psi \to \gamma) \to (\varphi \vee \psi \to \gamma) \\
(\mathrm{Ax7}) & (\varphi \to \psi) \wedge (\varphi \to \gamma) \to (\varphi \to \psi \wedge \gamma)
\end{array}
$$

$$
\begin{array}{ll}
\text{(Ax8)} & \varphi \leftrightarrow \neg\neg\varphi \\
\text{(Ax9)} & \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi) \\
\text{(Ax10)} & (\varphi \wedge \neg\varphi) \to (\neg(\psi \to \varphi) \to \gamma) \\
\text{(Ax11)} & (\varphi \to \psi) \to (\neg(\varphi \to \gamma) \to \psi) \\
\text{(Ax12)} & (\neg\varphi \to \psi) \to (\neg(\gamma \to \varphi) \to \psi) \\
\text{(Ax13)} & \bot \to \varphi \\
\text{(Ax14)} & (\varphi \wedge (\psi \to \bot)) \to \neg(\varphi \to \psi) \\
\text{(Ax15)} & (\varphi \wedge (\neg\varphi \to \bot)) \vee (\varphi \wedge \neg\varphi) \vee (\neg\varphi \wedge (\varphi \to \bot))
\end{array}
$$

where φ, ψ, γ are any well formed formulas and $\bot$ is the abbreviation for $\neg(\theta \to \theta)$ for any arbitrary formula θ.

Rules for LPS_3

The rules for LPS_3 are the following:

1. $$\frac{\varphi,\ \psi}{\varphi \wedge \psi}$$
2. $$\frac{\varphi,\ \varphi \to \psi}{\psi}$$

Soundness and completeness of LPS_3

Let $\vdash$ and $\models$ be the syntactic and semantic consequence relations respectively defined in the usual way with respect to the above mentioned axiom system and the matrix PS_3.

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Theorem

Soundness: For any formula φ and a set of formulas Γ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Theorem

Completeness: For any formula φ, if $\models \varphi$ then $\vdash \varphi$.

(Tarafder, S., & M. K. Chakraborty, A Paraconsistent Logic Obtained from an Algebra-Valued Model of Set Theory, to appear in: *New Directions in Paraconsistent Logic*, Springer, 2015.)

An Algebra-Valued Model of a Paraconsistent Set Theory

Hence we have reached to the fact that

$\mathbf{V}^{(\mathrm{PS}_3)}$ is an algebra-valued model of a paraconsistent set theory.

Some Classical Definitions in Metalanguage

Definition

A set x is said to be transitive if every element of x is a subset of x, or equivalently, if $y \in z$ and $z \in x$ implies $y \in x$.

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Definition

An ordinal number is a transitive set well-ordered by $\in$.

α-like Elements

For each $\alpha \in \mathrm{ORD}$ the α-like names in $\mathbf{V}^{(\mathrm{PS}_3)}$ are defined by transfinite recursion as follows.

Definition

An element $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ is called

1. 0-*like* if for every $y \in dom(x)$, we have that $x(y) = 0$; and
2. α-*like* if for each $\beta \in \alpha$ there exists $y \in dom(x)$ which is β-like and $x(y) \in \{1, 1/2\}$, and for any $z \in dom(x)$ if it is not β-like for any $\beta \in \alpha$ then $x(z) = 0$.

α-like Elements Meets Our Expectations

Theorem

Let $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ *be* α*-like for some* $\alpha \in \mathrm{ORD}$. *For any* $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![x = y]\!] = 1$ *if and only if* y *is* α*-like.*

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Theorem

Let $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ be α-like for some non-zero $\alpha \in \mathrm{ORD}$. For any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![y \in x]\!] \in \{1, 1/2\}$ if and only if y is β-like for some $\beta \in \alpha$.

Ordinals in First Order Language

As promised earlier, the definitions of transitive set, linear-ordered set, well-ordered set and ordinal number is written below in the set theoretic language.

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$$\mathrm{WO}_{\in}(x) = \mathrm{LO}(x) \wedge \forall y (y \subseteq x \wedge \neg (y = \varnothing) \rightarrow \exists z (z \in y \wedge z \cap y = \varnothing))$$

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$$\mathrm{ORD}(x) = \mathrm{Trans}(x) \wedge \mathrm{WO}_{\in}(x)$$

Ordinals in First Order Language

The following abbreviations are used in $\mathrm{WO}_{\in}(x)$:

$$y \subseteq x := \forall t(t \in y \to t \in x),$$

$$\neg(y = \varnothing) := \exists z(z \in y),$$

$$(z \cap y = \varnothing) := \neg \exists w(w \in z \wedge w \in y).$$

Some Results on Ordinal-Like Elements

Finally, we can connect the notion of α-like name to the set theoretic notion of ordinals:

Lemma

Let $\alpha \in \mathrm{ORD}$ and u be an α-like element in $\mathbf{V}^{(\mathrm{PS}_3)}$. Then the following hold:

1. $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Trans}(u)$
2. $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{LO}(u)$
3. $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{WO}_{\in}(u)$

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Hence we conclude the following theorem:

Theorem

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Hence we conclude the following theorem:

Theorem

Let $\alpha \in \mathrm{ORD}$ and u be an α-like element in $\mathbf{V}^{(\mathrm{PS}_3)}$. Then $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{ORD}(u)$.

Like the classical set theory we also have

Theorem

There is no set of all ordinals:

$$\mathbf{V}^{(\mathrm{PS}_3)} \not\models \exists O\, \forall x(\mathrm{ORD}(x) \rightarrow x \in O).$$

(Tarafder, S., Ordinals in an algebra-valued model of a paraconsistent set theory, *Logic and Its Applications*, LNCS, Vol. 8923. Berlin: Springer-Verlag, pp. 195–206, 2015.)

Leibniz's Law of the Indiscernibility of Identicals

It can be proved that Leibniz's law of the indiscernibility of identicals

$$\forall x \forall y (x = y \wedge \varphi(x) \to \varphi(y))$$

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is not valid in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all formula φ.

On the other hand it is also proved that for any instantiations of Leibniz's law with NFF-formulas φ is valid.

Paraconsistency in the Set Theory

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Let $\varphi := \exists x \exists y \exists z (x = y \wedge z \in x \wedge z \notin y)$. Then it can be proved that $[\![\varphi]\!] = 1/2$ and hence $[\![\neg\varphi]\!] = 1/2^* = 1/2$ which leads to the fact that both φ and $\neg\varphi$ are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

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4. The logic LPS_3 which is sound and complete with respect to PS_3 is a paraconsistent logic.
5. As a consequence, $\mathbf{V}^{(\mathrm{PS}_3)}$ is a model of some paraconsistent set theory.
6. Defined ordinal-like elements inside $\mathbf{V}^{(\mathrm{PS}_3)}$ and studied some classical and non-classical properties of them.

Thank You