PARACONSISTENCY: SOME BASIC ISSUES

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What is Paraconsistency?

Paraconsistent system of logic

A paraconsistent system of logic can be defined as a system that admits **inconsistent** but **non-trivial** theories.

Inconsistent Theory

A theory Γ is inconsistent iff there is a formula α (a sentence) such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \sim \alpha$.
i.e. α and negation of α both follow from Γ as premise.

Trivial Theory

A theory Γ is trivial iff $\Gamma \vdash \alpha$ for all wff α .
i.e. every wff follow from Γ.

What is Paraconsistency?

- In the classical logic Γ is inconsistent iff Γ is trivial.

  The main issue here is that any wff α can be derived from an inconsistent set Γ.

  This result is dependant on the fact that $\vdash \sim \alpha \to (\alpha \to \beta)$.
- We shall investigate later in some more detail the syntactic origin of the above equivalence.
- So the main objective of paraconsistent systems is to allow for $\Gamma \vdash \alpha$ and $\Gamma \vdash \sim \alpha$ for some Γ, α but form this not necessarily $\Gamma \vdash \beta$ for any β.

Vasiliev (1910)

proposed the ideal of a non-Aristotelian logic free of the laws of excluded middle and non-contradiction.
By analogy with the imaginary geometry of Lobachevsky, Vasiliev called his logic 'imaginary'.
This logic was not formalized.

Jaskowski (1949)

presented the first formal system for paraconsistent logic called 'discussive logic'.

History

Hallden (1949)

presented a 3-valued logic called ‘The logic of Nonsense’. This system can be considered as one of the first paraconsistent formal systems.

da Costa (1963)

presented his famous hierachy of paraconsistent systems $C_n (n \geqslant 1)$ constituting the broadest formal study of paraconsistency propsed till that time.

Asenjo (1966)

introduces a three-valued logic as a fromal framework for studying antinomies. This logic is structurally the same as that of Graham Priest.

Priest (1979)

The logic of paradox.

The expression "paraconsistent logic" was coined in a discussion between da Costa and Peruvian philosopher Francisco Miro Quesada in 1970.

Philosophy

- As mentioned before, paraconsistency is the study of logical systems in which the presence of contradiction does not imply triviality.

- What is the nature of contradiction?

  Ontological?

  Epistemological?

  Is reality intrinsically contradictory in the sense that we really need some pairs of contradictory propositions in order to describe it correctly?

  Or do contradictions have to do with knowledge or thought that have their origin in our cognitive apparatus, in the failure of measuring instruments, in the lack of appropriate language etc.?

Philosophy

- In the case of classical logic there is no situation (model) in which $\alpha \wedge \sim \alpha$ is satisfied (Law of non-contradiction LNC).

  and for all situations (models), $\alpha \vee \sim \alpha$ is satisfied (Law of excluded middle LEM).

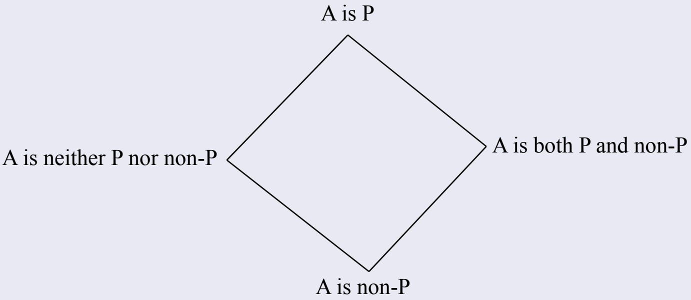
- Aristotle defends LNC, because in his view it cannot be the case that the same property belongs and does not belong to the same object. LNC is ontological.

  Similarly LEM is also ontological in the sense that given any property and an object the property either belongs or does not belong to the object.

- Given de-Morgan law and the law of double negation LNC and LEM are mutually obtainable, one from the other.

Indian Logic, Nagarjuna (50 A.D.- 120 A.D.):

CHATUSKOTI



Philosophy

- Intuitionistic logic intends to avoid improper use of LEM. It does not accept LEM.

  Does Intuitionistic logic give an account of truth preservation through its inference mechanism?

  We may say that it is about constructive truth, truth achieved in a constructive way.

  If we have a constructive proof of α

  we know that α is true,

  but the converse may not hold.

Philosophy

- Given an object A and a property P, the intuitionists will consider the claim

  'A is either P or non-P'

  as meaningless.

  They would be satisfied only when they know which one of the disjuncts is the fact.

- Is an intuitionist more inclined towards epistemic?

Van Dalen

"Two (logics) stand out as having a solid philosophical-mathematical justification. On one hand, classical logic with its ontological basis and on the other hand intuitionistic logic with its epistemic motivation."

Philosophy

- Paraconsistent logic does not accept LNC.
- That LNC can not be established as the nature of reality has been profusely discussed in the literature on paraconsistency.

  In formal sciences, e.g. mathematics: Russell's Paradox, Consistency of number theory.

  In empirical sciences, occurrences of contradiction in theories are abundant (c.f. da Costa & French, Meheus)

  However, there is no clear indication, far less a conclusive argument, that these contradictions are ontological and not only epistemological.

- Not accepting LNC from the ontological angle would mean that there are propositions α and $\sim \alpha$ such that both are true.
- Not accepting LNC from the epistemological angle may be interpreted as that there is evidence in favour of α and in favour of $\sim \alpha$.

Philosophy

- It is perfectly legitimate to devise formal systems in which contradictions are understood either ontologically or epistemologically.

  In the first case, it may be that both α and $\sim \alpha$ are true.

  In the second case we understand conflicting evidences for and against α.

  In the first case, rationality does not allow us to say that anything on earth <u>is true</u>.

  In the second case, one does not conclude any arbitrary assertion.

  Thus, in either case, existence of contradiction does not entail triviality.

- Non-acceptance of LNC is paraconsistency.

  Non-acceptance of LEM is paracompleteness.

mbc: a minimal LFI

Language is defined on the alphabet

$\{p_1, p_2, \ldots\}$

$\{0, \neg, \wedge, \vee, \rightarrow,), (\}$

Axioms

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
2. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
3. $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
4. $(\alpha \wedge \beta) \rightarrow \alpha$

Axioms

5. $(\alpha \wedge \beta) \rightarrow \beta$

6. $\alpha \rightarrow (\alpha \vee \beta)$

7. $\beta \rightarrow (\alpha \vee \beta)$

8. $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$

9. $\alpha \vee (\alpha \rightarrow \beta)$

10. $\alpha \vee \neg\alpha$

bc1. $o\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$

Rule

MP. $\frac{\alpha, \alpha \to \beta}{\beta}$

It can be shown that for some α, β

$\alpha, \neg\alpha \nvdash \beta, o\alpha, \alpha \nvdash, o\alpha, \neg\alpha \nvdash \beta.$

For all α, β

$o\alpha, \alpha, \neg\alpha \vdash \beta.$

mbc-valuation is a 2-valued mapping v, satisfying the following conditions.

$v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$

$v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$

$v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$

- $\star$ $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$
- $\star$ $v(o\alpha) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$

α	$\neg\alpha$
0	1
1	0/1

Because of this interpretation of negation, the system becomes paraconsistent but not paracomplete

α	$\alpha \vee \neg\alpha$
0	1
1	1

- v is a model for Γ iff $v(\gamma) = 1$ for all $\gamma \in \Gamma$.

  $\Gamma \models \alpha$ iff for every valuation v if v is a model for Γ, then v is also a model for α.

  One can establish soundness and completeness.

  $\Gamma \vdash \alpha$ iff $\Gamma \models \alpha$.

- Validity of Axiom bc1

α	0				1						
β	0		1		0			1			
$\neg\alpha$	1		1		0		1	0		1	
$o\alpha$	0	1	0	1	0	1	0	0	1	0	
$\neg\alpha \to \beta$	0	0	1	1	1	1	0	1	1	1	
$\alpha \to (\neg\alpha \to \beta)$	1	1	1	1	1	1	0	1	1	1	
$o\alpha \to (\alpha \to (\neg\alpha \to \beta))$	1	1	1	1	1	1	1	1	1	1	

3-valued matrices provide another semantic framework for introducing paracomsistent logics. In this framework there is no need to internalize the consistency operator.

Formulae obtain values in a 3-element set with the necessary algebraic structure.

The semantic consequence relation is defined with the help of a subset of designated values.

The Logic P_1 (Sette, 1973)

$\neg$	
1	0
$\frac{1}{2}$	1
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	0	0	0

$\alpha \vee \beta \equiv \neg(\neg(\alpha \wedge \alpha) \wedge \neg(\beta \wedge \beta))$

$\alpha \rightarrow \beta \equiv \neg((\beta \wedge \beta) \wedge \neg(\alpha \wedge \alpha))$

Designated set $\{1, \frac{1}{2}\}$

The Logic LP (Priest, 1979, Asenjo, 1966)

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

Designated set $\{1, \frac{1}{2}\}$

$\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$

System LPS_3 (Tarafdar, Chakraborty, 2015)

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	1	1	1

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

Designated set $\{1, \frac{1}{2}\}$

System LPS_3

- This is a complete distributive lattice satisfying conditions

  P_1: $x \wedge y \leq z$ implies $x \leq y \to z$

  P_2: $y \leq z$ implies $x \to y \leq x \to z$

  P_3: $y \leq z$ implies $z \to x \leq y \to x$

  P_4: $(x \wedge y) \to z = x \to (y \to z)$

System LPS_3

In the axiomatic system, there are 15 axioms and two rules,

$$\frac{\alpha,\beta}{\alpha\wedge\beta}$$

$$\frac{\alpha,\alpha\rightarrow\beta}{\beta}$$

The system is sound

$\Gamma \vdash \alpha$ implies $\Gamma \models \alpha$

The system is weakly complete

$\models \alpha$ implies $\vdash \alpha$

System LPS_3

- LPS_3 admits

  DN: $\vdash \neg\neg\alpha \leftrightarrow \alpha$

  DM_1: $\vdash \neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$

  DM_2: $\vdash \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$

  LEM: $\vdash \alpha \vee \neg\alpha$

  HS: $\vdash ((\alpha \to \beta) \wedge (\beta \to \gamma)) \to (\alpha \to \gamma)$

  MP: $\alpha, \alpha \to \beta \vdash \beta$

  DT: $\Gamma \cup \{\alpha\} \vdash \beta \Rightarrow \Gamma \vdash \alpha \to \beta$

There are several paraconsistent logics having algebraic semantics which are proposed from various motivations. Our motivation is to construct models of a paraconsistent set theory. From that angle the algebra proposed here is unique as shown below

	P_1	P_2	P_3	P_4
LPS_3	$\surd$	$\surd$	$\surd$	$\surd$
LP	$\times$	$\surd$	$\surd$	$\surd$
LF_1	$\times$	$\surd$	$\surd$	$\surd$
J_3	$\times$	$\surd$	$\surd$	$\surd$
RM_3	$\times$	$\surd$	$\surd$	$\times$

Consequence and Inconsistency

The notions of consequence, and hence inconsistency too, are interwoven in the context of classical logic. There are two sets of axioms characterising the notions of consequence and consistency. There is an equivalence between the two sets. Taking any set as primitive the other set can be obtained. This equivalence greatly depends on the fact that in the classical logic the notions of negation inconsistency and absolute inconsistency are equivalent.

$\Gamma \vdash \alpha, \neg\alpha$ iff $\Gamma \vdash \alpha$, for any α.

- Inconsistency in the context of paraconsistent logics is to be relativized. Let us call the notion para-inconsistency (PI).
- We say that

  $(\Gamma, \alpha) \in$ PI iff $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$,

  Γ is inconsistent w.r.t α.

  This does not imply that $(\Gamma, \beta) \in$ PI also, i.e. $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$ also.
- The notion was first introduced by Dutta and Chakraborty in 2011,

  A similar, but not the same, notion called α-contradictory set was introduced by Carnielli, Coniglio and Marcos in 2003.

Consequence and Inconsistency

We accept the structural rules

$\alpha \vdash \alpha$ (axiom)

$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$ (dilution right)

$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$ (dilution left)

$\frac{\alpha, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\alpha}$ (¬-right)

$\frac{\Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta}$ (¬-left)

One obtains

$\alpha \vdash \neg\neg\alpha$ (¬¬-I rule)

$\neg\neg\alpha \vdash \alpha$ (¬¬-E rule)

Consequence and Inconsistency

- ¬-left + dilution right ⇒ explosion

$$\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha, \neg\alpha \vdash}}{\alpha, \neg\alpha \vdash \beta}$$

- ¬-right + ¬-left + dil-left + cut ⇒ explosion.
- So to avoid explosion not to take

  ¬-left and dil together

  and ¬-right, ¬-left and dil together.
- $C(\Gamma) = \{\alpha | \Gamma \vdash \alpha\}$.

Consequence and Inconsistency

We propose notion of non-explosive consequence:

C: $\mathcal{P}(F) \to \mathcal{P}(F)$ is a consequence operator that has to satisfy the following conditions

C_1: $\Gamma \subseteq C(\Gamma)$

C_2: $\Gamma \subseteq \Delta \Rightarrow C(\Gamma) \subseteq C(\Delta)$

C_3: $CC(\Gamma) = C(\Gamma)$

C_4: $C(\{\alpha, \neg\alpha\}) \neq F$ for some α

C_5: $C(\Gamma \cup \{\alpha\}) \cap C(\Gamma \cup \{\neg\alpha\}) = C(\Gamma)$

C_6: $C(\{\alpha\}) = C(\{\neg\neg\alpha\})$.

Para-inconsistency axioms:

PI $\subseteq \mathcal{P}(F) \times$ F satisfying:

PI_1: $(\Gamma \cup \{\neg\alpha\}, \alpha) \in$ PI for all Γ, α

PI_2: $\Gamma \subseteq \Delta$ and $(\Gamma, \alpha) \in$ PI $\Rightarrow (\Delta, \alpha) \in$ PI

PI_3: If for all $\delta \in \Delta$, $(\Gamma \cup \{\neg\delta\}, \delta) \in$ PI and $(\Gamma \cup \Delta, \alpha) \in$ PI then (Γ, α) $\in$ PI

PI_4: There exist α, β such that $(\{\alpha, \neg\alpha\}, \beta) \notin$ PI

PI_5: $(\Gamma \cup \{\alpha\}, \beta) \in$ PI and $(\Gamma \cup \{\neg\alpha\}, \beta) \in$ PI $\Rightarrow (\Gamma, \beta) \in$ PI

PI_6: $(\Gamma, \neg\alpha) \in$ PI $\Rightarrow (\Gamma, \alpha) \in$ PI

PI_7: $(\Gamma \cup \{\neg\neg\alpha\}, \beta) \in$ PI $\Rightarrow (\Gamma \cup \{\alpha\}, \beta) \in$ PI.

Consequence and Inconsistency

Theorem

Let PI be given.
Let C be defined as follows:
$\alpha \in C(\Gamma)$ iff $(\Gamma \cup \{\neg\alpha\}, \alpha) \in PI$.
Then C satisfies conditions $C_1 - C_6$.

Theorem

Let C be given.
Let PI be defined as follows:
$(\Gamma, \alpha) \in PI$ iff $\{\alpha, \neg\alpha\} \subseteq C(\Gamma)$.
Then PI satisfies conditions $PI_1 - PI_7$.

Some of the paraconsistent logics satisfying the consequence axioms and hence inconsistency axioms too are the following:

1. D_2 (discussive logic, Jaskowski)
2. J_n $1 \leqslant n \leqslant 5$, (Arruda, da Costa, 1968)
3. J_3 (da Costa, D$'$Ottavino, 1970)
4. Calculus of autinomies (Asenjo, 1966)
5. LP (Logic of paradox, Priest, 1979)
6. Pac (Avron, 1991)
7. Cie systems (Carnelli, coniglio, Marcos, 2003)

- There are two versions of explosiveness

  (i) $\{\alpha, \neg\alpha\} \vdash \beta$ for all α, β [explosion]

  (ii) $\alpha \& \neg\alpha \vdash \beta$ for all α, β [&-explosion]

  (i) and (ii) are not necessarily the same.

  Systems where only (i) is violated is called weakly paraconsistent.

  Systems where both (i) and (ii) are violated is called strongly paraconsistent systems.

Consequence and Inconsistency

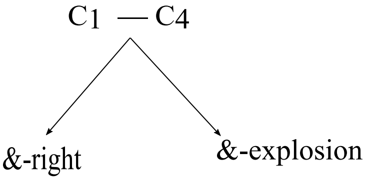
- The rule &-right is the following

$$\frac{\Gamma \vdash \alpha,\, \Gamma \vdash \beta}{\Gamma \vdash \alpha \& \beta}$$

- In presence of $C_1 - C_4$

  if &-right holds then &-explosion does not hold and if &-explosion holds then &-right does not hold.

  Hence the paraconsistent systems bifarcate.



Concluding Remarks

1. In the literature of paraconsistency, we see that various approaches are taken to design non-explosive consequence. But the notion of inconsistency has not been modified accordingly to match such inconsistency tolerant notion of consequences. We expect that this relativized notion of inconsistency and its interrelation with non-explosive consequence may help in getting alternative routes for proving metatheorems of paraconsistent systems.

Concluding Remarks

2. We know that intuitionistic or constructivist mathematics has developed to a great extent.

It is now time to develop paraconsistent mathematics. There has been several (not many) attempts in this direction.

Once mathematics is developed, applications would follow. Of course fuzzy mathematical techniques may be considered to fall within paraconsistency in wide sense since the LNC does not hold in fuzzy logics.

However, a full set theory and mathematics on it, both formal and informal, is the need of the time.

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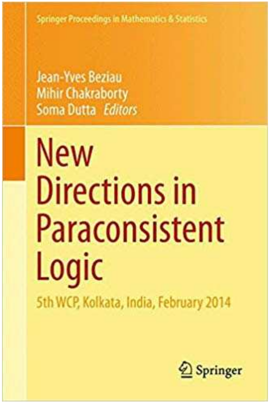

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Thank You